

On Blowup in Nonlinear Heat Equations*

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Abstract

We establish the asymptotics of blowup for nonlinear heat equations with superlinear power nonlinearities in arbitrary dimensions and we estimate the remainders.

1 Introduction

In this paper we study the blowup problem for the n -dimensional nonlinear heat equation (or the reaction-diffusion equation)

$$\begin{cases} \partial_t u &= \Delta u + |u|^{p-1}u \\ u(x, 0) &= u_0(x) \end{cases} \quad (1)$$

with $p > 1$. Here $u : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}$. Eq. (1) arises in the problem of heat flow, or, more generally, in the problems involving diffusion, and is a model for a large class of nonlinear parabolic equations, which are ubiquitous in mathematics and its applications.

We will deal, without mentioning it, with weak solutions of Eq. (1) in the sense detailed in the next section. The local existence of such solutions is well known (see, e.g. [1] for the Sobolev spaces H^α , $0 \leq \alpha < 2$) and is presented for readers' convenience in the next section. These solutions can be shown to be classical for $t > 0$.

For some data $u_0(x)$, the solutions $u(x, t)$ might blow up in finite time $t^* > 0$, i.e. they exist in L^∞ for $[0, t^*)$ and $\sup_x |u(x, t)| \rightarrow \infty$ as $t \rightarrow t^*$. Thus, two key problems about (1) are

1. Describe initial conditions for which solutions of Eq. (1) blow up in finite time;
2. Describe the blowup profile of such solutions.

It is expected (see e.g. [2]) that the (stable) blowup profile is universal—it is independent of lower power perturbations of the nonlinearity and of initial conditions within certain spaces.

The following key properties of equation (1) elucidate important features of the results we discuss below:

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- (1) is invariant with respect to the scaling transformation,

$$u(x, t) \rightarrow \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t) \quad (2)$$

for any constant $\lambda > 0$, i.e. if $u(x, t)$ is a solution, so is $\lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t)$.

- (1) has x -independent (homogeneous) solutions:

$$u_{\text{hom}} = [u_0^{-p+1} - (p-1)t]^{-\frac{1}{p-1}}. \quad (3)$$

Solutions (3) blow up in finite time $t^* = \left((p-1)u_0^{p-1}\right)^{-1}$ for $p > 1$. The linearization of (1) around u_{hom} shows that the solution u_{hom} is unstable. Moreover, it is shown in [18] that if either $n \leq 2$ or $p \leq (n+2)/(n-2)$, then the equation (1) has no other self-similar solutions of the form $(T-t)^{-\frac{1}{p-1}} \phi(x/\sqrt{T-t})$, $\phi \in L^\infty$, besides u_{hom} .

We consider (1) with initial conditions in a certain neighbourhood of the homogeneous solution, which have, modulo a small perturbation, a maximum at the origin, are slowly varying near the origin and are sufficiently small, but not necessarily vanishing, for large $|x|$. We show that the solutions of (1) for such initial conditions blowup at a finite time t^* and at some moving point $\zeta(t)$ and we characterize the blowup asymptotics. More precisely, with the standard notation $yby := \sum_{i,j=1}^n y_i b_{ij} y_j$ for a $n \times n$ -matrix $b := (b_{ij})$, $\langle x \rangle := (1 + |x|^2)^{1/2}$ and $f \lesssim g$ for two positive functions f and g , satisfying $f \leq Cg$ for some universal constant C , we have the following result.

Theorem 1. *Let $b_0 := (b_{0ij}) > 0$ be a real, symmetric, positive $n \times n$ -matrix with $\|b_0\| \ll 1$ and $1 \leq c_0 \leq 4$. Suppose the initial data $u_0 \in L^\infty(\mathbb{R}^n)$ for (1) satisfy the conditions*

$$\left\| \langle x \rangle^{-m} \left(u_0(x) - \left(\frac{c_0}{p-1 + x b_0 x} \right)^{\frac{1}{p-1}} \right) \right\|_\infty \leq \delta_m, \quad (4)$$

with $m = 0, 3$, $0 \leq \delta_0 \ll 1$ and $\delta_3 = C\|b_0\|^2$. Then

- (1) There exists a time $t^* \in (0, \infty)$ such that the solution $u(x, t)$ exists on the interval $[0, t^*)$ and blows up at t^* .
- (2) For $t < t^*$ there exist unique, C^1 , positive, real valued functions $\lambda(t)$ and $c(t)$, a C^1 n -vector valued function $\zeta(t)$, and a C^1 $n \times n$ -symmetric-matrix valued function $b(t)$, with $b(t) \lesssim b(0)$, such that $u(x, t)$ can be written as

$$u(x, t) = \lambda^{\frac{2}{p-1}}(t) \left[\left(\frac{c(t)}{p-1 + y b(t) y} \right)^{\frac{1}{p-1}} + \xi(x, t) \right], \quad (5)$$

where $y := \lambda(t)(x - \zeta(t))$ and the fluctuation part, ξ , admits the estimates $\|\langle y \rangle^{-m} \xi(x, t)\|_\infty \lesssim \delta_m(t)$, $m = 0, 3$. Here $\delta_0(t) = \delta_0 \ll 1$ and $\delta_3(t) = \|b(t)\|^2$.

- (3) The parameters $\lambda(t)$, $b(t)$, $c(t)$ and $\zeta(t)$ obey certain dynamical equations (with initial conditions $\lambda(0) = \sqrt{c_0 + \frac{2}{p-1} \text{Tr } b(0)}$, $c_0 > 0$, $b(0) > 0$, depending on the initial datum), whose solutions give

$$\begin{aligned} \lambda(t) &= (t^* - t)^{-\frac{1}{2}}(1 + o(1)) \\ b(t) &= \frac{(p-1)^2}{4p |\ln |t^* - t||} (I + O(\frac{1}{|\ln |t^* - t||^{1/2}})) \\ c(t) &= 1 - \frac{p-1}{2p |\ln |t^* - t||} (1 + O(\frac{1}{|\ln |t^* - t||})) \\ \zeta(t) &= O(1). \end{aligned} \quad (6)$$

Here $o(1)$ is in $t^* - t$.

Remarks. 1) Neither smoothness of initial conditions nor decay at infinity are required. In particular, the energy

$$\mathcal{E}(u) := \int \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |u|^{p+1} \right) dx, \quad (7)$$

for such initial conditions might be infinite.

2) The weight in the L^∞ -norm for $\xi(x, t)$ comes from the fact that we decompose a solution of (1) into a leading profile and a fluctuation, with the fluctuation orthogonal to bad (positive or nearly zero) eigenvalues of the linearization around the leading profile. The weight in question is determined by the eigenfunction of the first good (negative) eigenvalue. (Note that bad eigenvalues reflect the instabilities w.r. to the blowup time and center and the shape and size of the blowup profile.)

There is rich literature regarding the blowup problem for equation (1). We review quickly the relevant results. Starting with [16], various criteria for blowup in finite time were derived, see e.g. [16, 1, 5, 8, 27, 28, 37, 39, 43, 11, 15]. For example, if $u_0 \in H^1 \cap L^{p+1}$ and $\mathcal{E}(u_0) < 0$, then it is proved in [27] that $\|u(t)\|_2^2$ blows up in finite time t^* . By the observation

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 \leq \|u(t)\|_\infty^{p-1} \|u(t)\|_2^2$$

we have that $\|u(t)\|_\infty$ blows up in finite time $t^{**} \leq t^*$ also. (In this paper, we denote the norms in the L^p spaces by $\|\cdot\|_p$.)

Blowup at a single point was studied as early as [46] (see also [15]). The first result on asymptotics of the blowup for arbitrary dimension $n \geq 1$ was obtained in the pioneering paper [18] where the authors show that under the condition

$$|u(x, t)|(t_* - t)^{\frac{1}{p-1}} \text{ is bounded on } B_1 \times (0, t_*), \quad (8)$$

where B_1 is the unit ball in \mathbb{R}^n centred at the origin, and either $p \leq \frac{n+2}{n-2}$ or $n \leq 2$ and assuming blowup takes place at $x = 0$, one has

$$\lim_{\lambda \rightarrow 0} \lambda^{\frac{2}{p-1}} u(\lambda x, t_* + \lambda^2(t - t_*)) = \pm \left(\frac{1}{p-1} \right)^{\frac{1}{p-1}} (t_* - t)^{-\frac{1}{p-1}} \text{ or } 0.$$

This result was further improved in several papers (see e.g. [20, 19, 23, 12, 30, 44, 13, 14, 15, 2, 33, 34, 35]). A blowup solution satisfying the bound (8) is said to be of type I. This bound was proven under various conditions in [20, 33, 34, 47, 21]. Furthermore, the limits of H^1 -blowup solutions $u(x, t)$ as $t \uparrow T$, outside the blowup sets were established in [23, 12, 30, 44, 13, 14, 15, 2, 35, 9].

For $p > 1$, dimension $n = 1$, Herrero and Velázquez [24] (see also [13]) proved that if the initial condition u_0 is continuous, nonnegative, bounded, even, and has only one local maximum at 0, and if the corresponding solution blows up, then

$$\lim_{t \uparrow t^*} (t^* - t)^{\frac{1}{p-1}} u(y((t^* - t) \ln |t^* - t|)^{\frac{1}{2}}, t) = (p-1)^{-\frac{1}{p-1}} \left[1 + \frac{p-1}{4p} y^2 \right]^{-\frac{1}{p-1}}, \quad (9)$$

uniformly on sets $|y| \leq R$ with $R > 0$. Further extensions of this result are achieved in [23, 44, 12, 13].

Later for dimension $n = 1$ Brimont and Kupiainen [2] constructed a co-dimension 2 submanifold of initial conditions such that (9) is satisfied on the whole domain. More specifically, given a small function g and a small constant $b > 0$, they find constants d_0 and d_1 depending on g and b such that the solution to (1) with the datum

$$u_0^*(x) = (p - 1 + bx^2)^{-\frac{1}{p-1}} \left(1 + \frac{d_0 + d_1 x}{p - 1 + bx^2}\right)^{\frac{1}{p-1}} + g(x) \quad (10)$$

has the convergence (9) uniformly in $y \in (-\infty, +\infty)$. The result of [2] was generalized in [32] (see also [17]), where it is shown that there exists a neighborhood \mathcal{U} , in the space $L^{p+1} \cap H^1$, of u_0^* , given in (10), such that if $u_0 \in \mathcal{U}$, then the solution $u(x, t)$ blows up in a finite time t^* and satisfies (9) for $x \in \mathbb{R}$. They conjectured that this asymptotic behavior is generic for any blowup solution.

For initial conditions in L^∞ that lead to blowup at a prescribed location and time, a, T , respectively, with the blowup profile (9), Merle et al. ([9, 30, 32, 33, 34, 35]) established the stability of the blowup profile in any dimension.

In [6] precise blowup asymptotics were derived for (1) in dimension 1 for even initial conditions. Our results extend the results of [6] in two aspects. First, we address the problem of blowup in arbitrary dimensions. Second, we consider more general, non-symmetric initial conditions, which allow the blowup center to move.

F. Merle ([31]) has informed the last author that asymptotics (5) - (6), but without estimates of the remainders, can be derived from [9, 10, 20, 21, 29, 36, 45].

Unlike the most of the works above, we do not use the fact that (1) is an L^2 -gradient system

$$\partial_t u = -\text{grad } \mathcal{E}(u),$$

with the energy defined in (7). Instead we use method of majorants, which allow us to bootstrap our estimates, and strong linear estimates. Hence we expect our analysis can be extended to non-gradient systems.

Also, in contrast all previous works, with exception of [6], which fix scaling as $\lambda(t) = (T - t)^{-\frac{1}{2}}$, where T is the blowup time, we leave the scaling, $\lambda(t)$ (and blowup center, shape and size parameters, b and c , and time) to be determined by the equation. As a result we obtain a dynamical system for the scaling parameter $\lambda(t)$, as well as for other parameters determining the leading profile, solving which gives the desired scaling law. Hence our approach is well adapted to detecting the scaling dynamics in situations where scaling law is not obvious (see e.g. [53, 52, 48, 49, 50, 51]).

We believe our techniques are sufficiently simple and robust and can more or less straightforwardly be extended to $p < 0$ (collapse, see [54]), to several blowup centers, to blowups along spheres and to more general, say polynomial, nonlinearities.

Our proof is close to the one of [6] but several points are substantially revised and the exposition is simplified. Since the problem is important and our treatment is still simpler than anything presented so far in the literature, we give, for the reader's convenience a complete proof, reproducing some of the results of [6].

Our approach consists of the following sequence of steps:

- Passing to blowup variables (Section 2). Given differentiable functions $z(t) \in \mathbb{R}^n$ and $\lambda(t) > 0$, we pass

to new variables as

$$v(y, \tau) := \lambda^{-\frac{2}{p-1}}(t)u(x, t), \quad \text{where} \quad y := \lambda(t)(x - z(t)) - \alpha(t) \quad \text{and} \quad \tau := \int_0^t \lambda^2(s)ds.$$

Here $\alpha(t)$ satisfies the equation $\lambda^{-2}\dot{\alpha} - a\alpha = -\lambda^{-1}\dot{z}$, with $a(t) = \dot{\lambda}(t)/\lambda^3(t)$. Now $\lambda(t), a(t), z(t)$ and $v(y, \tau)$ are unknowns we have to solve for.

- Reparametrization of solutions (Section 3). The equation for $v(y, \tau)$, which follows from (1), has the two-parameter family of approximate solutions

$$v_{ab} := \left(\frac{2a}{p-1+yby} \right)^{\frac{1}{p-1}}, \quad (11)$$

where $b := (b_{ij})$, $b_{ij} \in \mathbb{R}$, is any real, symmetric $n \times n$ -matrix and, recall, $yby := \sum_{i,j=1}^n y_i b_{ij} y_j$:

$$\left(\Delta - ay \cdot \nabla - \frac{2a}{p-1} \right) v_{ab} + |v_{ab}|^{p-1} v_{ab} \approx 0. \quad (12)$$

In what follows we take $b \geq 0$, so that v_{ab} is nonsingular.

It will turn out (see below in this outline) that a approaches $1/2$, as t approaches the blowup time, and it will be convenient to replace v_{ab} by $V_{ab}(y) := \left(\frac{a+1/2}{p-1+yby} \right)^{\frac{1}{p-1}}$. We consider the manifold

$$\mathcal{M}_{as} := \{V_{ab} \mid a \in \mathbb{R}_+, b \in \mathbb{R}^{n \times n}\}$$

of almost solutions. We parameterize a solution by a point on the manifold \mathcal{M}_{as} and a fluctuation (approximately) orthogonal to this manifold:

$$v = V_{ab} + \xi, \quad \xi \perp T_{V_{ab}} \mathcal{M}_{as}, \quad (13)$$

in the sense of $L^2(\mathbb{R}^n, e^{-a|y|^2/2} dy)$ (large slow moving and small fast moving parts of the solution).

- Lyapunov-Schmidt decomposition (Section 6). Plugging the decomposition (13) into the equation for $v(y, \tau)$ gives the equation

$$\xi_\tau = -\mathcal{L}_{ab}\xi + \mathcal{N}(\xi, a, b) + \mathcal{F}(a, b) \quad (14)$$

where \mathcal{L}_{ab} , $\mathcal{N}(\xi, a, b)$ and $\mathcal{F}(a, b)$ are the linear operator, the nonlinearity and the source term respectively.

Differentiating the equation (12) w.r. to a , z (remember, $y := \lambda(t)(x - z(t))$) and b , and using that $V_{ab} = (p-1)a\partial_a V_{ab}$ and $y \cdot \nabla V_{ab} = \frac{1}{p-1} \frac{2yby}{p-1+yby} V_{ab} = \frac{2ayby}{p-1+yby} \partial_a V_{ab}$ and $\nabla V_{ab} = \lambda^{-1} \nabla_z V_{ab}$, we obtain

$$\mathcal{L}_{ab}(\partial_a V_{ab}) \approx a(1 + \frac{yby}{p-1+yby}) \partial_a V_{ab}, \quad \mathcal{L}_{ab}(\nabla_z V_{ab}) \approx a \nabla_z V_{ab}, \quad \mathcal{L}_{ab}(\partial_{b_{ij}} V_{ab}) \approx 0. \quad (15)$$

Since for $|y|$ bounded and $\|b\|$ small, $V_{ab} \approx \left(\frac{a+1/2}{p-1} \right)^{\frac{\mu}{p-1}}$ and therefore

$$\partial_a V_{ab} \approx \frac{1}{a}, \quad \nabla_{z_j} V_{ab} \approx \lambda \mu \sum_j b_{ij} y_j, \quad \text{and} \quad \partial_{b_{ij}} V_{ab} \approx \mu y_i y_j,$$

where $\mu := \frac{1}{p-1} \left(\frac{a+1/2}{p-1} \right)^{\frac{1}{p-1}}$. Hence we expect that the linearized operator \mathcal{L}_{ab} has approximate eigenvalues $2a$, a and 0 with the corresponding approximate eigenfunctions of 1 , y_j and $y_i y_j$, which are approximate tangent vectors to \mathcal{M}_{as} at V_{ab} spanning $T_{V_{ab}} \mathcal{M}_{\text{as}}$.

The first two groups of approximate eigenfunctions are related to the scaling and translation symmetry of the original nonlinear heat equation (1). The third one can be thought of as related to the symmetry w.r. to rotations. The approximate eigenfunctions above give the unstable modes in our problem and they will play an important role in our analysis.

Consider now the family $\tilde{V}_{bc}(y) := \left(\frac{c}{p-1+by} \right)^{\frac{1}{p-1}}$, with c an extra parameter, and proceed as above, using the decomposition $v = \tilde{V}_{bc} + \xi$, instead of (13). Projecting the resulting equation for ξ onto approximate $T_{\tilde{V}_{bc}} \mathcal{M}_{\text{as}}$, we find the following dynamical system for the parameters a, b, c :

$$\partial_\tau c = c(c-2a) - \frac{2}{p-1} \text{Tr } b + \text{Rem}_c(\xi, a, b, c) \quad (16)$$

$$\partial_\tau b = (c-2a)b - \frac{2b}{p-1} \text{Tr } b + \frac{4p}{(p-1)^2} b^2 + \text{Rem}_b(\xi, a, b, c), \quad (17)$$

for some remainders $\text{Rem}_c(\xi, a, b, c)$ and $\text{Rem}_b(\xi, a, b, c)$ which are expected to provide higher order corrections. Note that we are free to choose the (time-dependent) additional parameter c at our convenience. From the above equations we read off the equilibria (zeroes of the vector field governing the evolution of the parameters a, b, c) as

$$(a, b, c) = (a^*, 0, 2a^*),$$

for any choice of function a^* . The fixed point we want the parameters to flow to is

$$(a, b, c) = \left(\frac{1}{2}, 0, 1 \right).$$

One way to achieve this is to fix c as a convex combination of 1 and $2a$: $c = \rho + 2(1-\rho)a$, for any $\rho \in (0, 1)$. Note that the extremal point $\rho = 0$ is not a good choice because the equation for c_τ would lose its leading part driving a and c to the desired fixed point, while $\rho = 1$ robs us of an equation for a_τ . The simplest choice is $\rho = \frac{1}{2}$, so that

$$c = \frac{1}{2} + a \quad \text{and} \quad c - 2a = \frac{1}{2} - a.$$

This is exactly our reason for using $V_{ab}(y) := \left(\frac{a+1/2}{p-1+by} \right)^{\frac{1}{p-1}}$, instead of $v_{ab} := \left(\frac{2a}{p-1+by} \right)^{\frac{1}{p-1}}$.

- Linear propagator estimates (Section 9). Using combination of techniques we derive estimates of the propagators generated by the operator \mathcal{L}_{ab} in the norms introduced above.
- Majorants and bootstrap (Sections 4, 7, 10, 11). To control the fluctuations $\xi(\tau)$, we introduce the estimating functions (families of semi-norms)

$$M_k(T) := \max_{\tau \leq T} \beta^{-2(2-k)}(\tau) \|\langle y \rangle^{-3(2-k)} \xi(\tau)\|_\infty, \quad k = 1, 2,$$

and similarly for the parameters $b(\tau)$ and $a(\tau)$. Using (14) and the linear propagator estimates, we prove inequalities for these estimating functions, which allow us to bootstrap our estimates, starting from very rough ones provided by the local well-posedness. This allows us to propagate our estimates in time.

We conclude the introduction by stating without proof the standard result on the local well-posedness of (1). $W^s := \{u \in L^\infty, (-\Delta)^{s/2}u \in L^\infty\}$.

Theorem 2. *Let $u_0 \in L^\infty$. Then there exists t_* such that*

- (1) has a unique mild solution in $C([0, t_*), L^\infty)$;
- u depends continuously on the initial condition u_0 ;
- Either $t_* = \infty$ or $t_* < \infty$ and $\|u(t)\|_\infty \rightarrow \infty$ as $t \rightarrow t_*$;
- If $u_0 \in W^s$, $s \geq 0$, then $\|\partial_t u\|_\infty \lesssim t^{-\max(1-\frac{s}{2}, 0)}$ as $t \rightarrow 0$. In particular, $u \in C^1((0, t_*), L^\infty)$ ($C^1([0, t_*), L^\infty)$ if $s \geq 2$).

2 Blowup Variables and Almost Solutions

Let $z(t) \in \mathbb{R}^n$, $\lambda(t) > 0$, be differentiable functions and let $\alpha(t)$ satisfy the equation

$$\lambda^{-2}\dot{\alpha} - a\alpha = -\lambda^{-1}\dot{z}, \quad (18)$$

with $a(t) = \dot{\lambda}(t)/\lambda^3(t)$. We introduce the blowup variables

$$y := \lambda(t)(x - z(t)) - \alpha(t) \text{ and } \tau := \int_0^t \lambda^2(s) ds$$

and define the new function

$$v(y, \tau) := \lambda^{-\frac{2}{p-1}}(t)u(x, t). \quad (19)$$

Plugging (19) into (1) we obtain

$$\partial_\tau v = \left(\Delta_y - ay \cdot \nabla_y - \frac{2a}{p-1} \right) v + |v|^{p-1}v, \quad (20)$$

where, as above, $a(t) = \dot{\lambda}(t)/\lambda^3(t)$. The initial condition for this equation is obtained from the initial condition for (1) as $v(y, 0) = \lambda_0^{-\frac{2}{p-1}}u_0(z_0 + \frac{y+\alpha_0}{\lambda_0})$, for some λ_0, z_0 and α_0 .

From the local well-posedness of (1) and using rescaling, we can conclude that there exists $T > 0$ s.t. (20) has a unique mild solution in $C([0, T], L^\infty)$ and the solution depends continuously on the initial condition. Moreover, either $T = \infty$ or $T < \infty$ and $\|v(\tau)\|_\infty \rightarrow \infty$ as $\tau \rightarrow T$.

The equation (20) has the following family of homogeneous, static (i.e. y and τ -independent) solutions: a is a constant and

$$v_a := \left(\frac{2a}{p-1} \right)^{\frac{1}{p-1}}. \quad (21)$$

This family of solutions corresponds to the homogeneous solution (3) of the nonlinear heat equation with the parabolic scaling $\lambda^{-2} = 2a(T - t)$, where the blowup time, $T := \left[u_0^{p-1}(p-1) \right]^{-1}$, is dependent on the initial value, u_0 of the homogeneous solution $u_{\text{hom}}(t)$.

If the parameter a is τ dependent but $|a_\tau|$ is small, then the above solutions are good approximations to the exact solutions. A richer family of approximate solutions is obtained by solving the equation $ay \cdot \nabla_y v + \frac{2a}{p-1}v = v^p$, obtained from (20) by neglecting the τ derivative and second order partial derivative in y . This equation has the general solution

$$v_{ab}(y) := \left(\frac{2a}{p-1+by} \right)^{\frac{1}{p-1}} \quad (22)$$

for all $b := (b_{ij}), b_{ij} \in \mathbb{R}$, real, symmetric $n \times n$ -matrices. Here recall $yby := \sum_{i,j=1}^n y_i b_{ij} y_j$. In what follows we take $b \geq 0$, so that v_{ab} is nonsingular. Note that $v_{2a,0} = v_a$.

3 Reparametrization of Solutions

In this section we split solutions to (20) into the leading term—the almost solution $V_{ab}(y) := \left(\frac{a+1/2}{p-1+by} \right)^{\frac{1}{p-1}}$ —and a fluctuation ξ around it. (The reason for passing from $v_{ab}(y) := \left(\frac{2a}{p-1+by} \right)^{\frac{1}{p-1}}$ to $V_{ab}(y)$ was explained in the introduction.) More precisely, we would like to parameterize a solution by a point on the manifold $\mathcal{M}_{\text{as}} := \{V_{ab} \mid a \in \mathbb{R}_+, b \in \mathbb{R}^{n \times n}\}$ of almost solutions and the fluctuation orthogonal to this manifold (large slow moving and small fast moving parts of the solution). For technical reasons, it is more convenient to require the fluctuation to be almost orthogonal to the manifold \mathcal{M}_{as} . More precisely, recalling the discussion at the end of the previous section, we require ξ to be orthogonal to the vectors $\phi_a^{(ij)}, 0 \leq i, j \leq n$, where

$$\phi_a^{(00)}(y) := 1, \phi_a^{(0i)}(y) = \phi_a^{(i0)}(y) := \sqrt{a}y_i, \phi_a^{(ij)}(y) := ay_i y_j, 1 \leq i, j \leq n,$$

which are almost tangent vectors to the above manifold, provided b is sufficiently small.

Denote by \mathbb{M}_n the space of real, symmetric, $n \times n$ matrices and by \mathbb{M}_n^+ , the positive cone in this space. Let $u_{\lambda,z}(y) := \lambda^{-\frac{2}{p-1}}u(x)$, with $x = z + \lambda^{-1}(y + \alpha)$. We define the neighborhoods

$$U_\epsilon := \{v \in L^\infty(\mathbb{R}^n) \mid \|e^{-\frac{1}{3}|y|^2}(v - V_{ab})\|_\infty \leq C\|b\|^2 \text{ for some } 1/4 \leq a \leq 1, 0 < b \leq \epsilon\}$$

and

$$\tilde{U}_\epsilon := \{u \in L^\infty(\mathbb{R}^n) \mid u_{\lambda,z} \in U_\epsilon\}.$$

The following statement will be used to reparametrize the initial conditions.

Proposition 3. *There exist an $\epsilon_0 > 0$ and a unique C^1 functional $g : U_{\epsilon_0} \rightarrow \mathbb{R}^+ \times \mathbb{M}_n^+ \times \mathbb{R}^n$, such that any function $u_{\lambda,z_0} \in U_{\epsilon_0}$ can be uniquely written in the form*

$$u_{\lambda,z_0} = V_{ab} + \xi, \quad (23)$$

with $\xi \perp \phi_a^{(ij)}, 0 \leq i, j \leq n$, in $L^2(\mathbb{R}^n, e^{-\frac{a|y|^2}{2}} dy)$, $(a, b, z) = g(u_{\lambda,z_0})$. Moreover, if $\frac{1}{4} \leq a_0 \leq 1, 0 < b_0 \leq \epsilon_0$ and $\|\langle y \rangle^{-m}(u_{\lambda,z_0} - V_{a_0 b_0})\|_\infty \leq \delta_m$ with $m = 0, 3$, $\delta_3 = O(\|b_0\|^2)$ and δ_0 small, we have

$$|g_1(u_{\lambda,z_0}) - (a_0, b_0)| \lesssim \|b_0\|^2, \quad (24)$$

$$|g_2(u_{\lambda, z_0}) - z_0| \lesssim \|b_0\|, \quad (25)$$

$$\|\langle y \rangle^{-3}(u_{\lambda, z_0} - V_{g(u_{\lambda, z_0})})\|_\infty \lesssim \|b_0\|^2, \quad (26)$$

$$\|u_{\lambda, z_0} - V_{g(u_{\lambda, z_0})}\|_\infty \lesssim \delta_0 + \|b_0\|. \quad (27)$$

for $g(u_{\lambda, z_0}) = (g_1(u_{\lambda, z_0}), g_2(u_{\lambda, z_0}))$, where $g_1(u_{\lambda, z_0}) = (a, b)$ and $g_2(u_{\lambda, z_0}) = z$.

Proof. Let $V_{\lambda abz}(x) := \lambda^{\frac{2}{p-1}} V_{ab}(y)$, $V_\mu \equiv V_{\lambda abz}$ with $\mu = (a, b, z)$, and $\varphi_{az}^{(ij)}(x) := \phi_a^{(ij)}(y)$ with $y := \lambda(x - z) - \alpha$. The orthogonality conditions on the fluctuation can be written as $G(\mu, u) = 0$, where $G : \mathbb{R}^+ \times \mathbb{M}_n^+ \times \mathbb{R}^n \times L^\infty(\mathbb{R}^n) \rightarrow \mathbb{M}_{n+1}$ is defined as

$$G(\mu, u) := \left(\left\langle V_\mu - u, \varphi_{az}^{(ij)} \right\rangle \right).$$

Here and in what follows, all inner products are $L^2(\mathbb{R}^n, e^{-a|y|^2/2} dy)$ inner products. Whenever it is convenient we identify μ with an $(n+1) \times (n+1)$ -matrix: $\mu_{00} := a$, $\mu_{0i} = \mu_{i0} = z_i$, $\mu_{ij} := b_{ij}$, $1 \leq i, j \leq n$ and let $\mathbb{M}_{n+1}^{++} := \{\mu \in \mathbb{M}_{n+1} \mid a \geq 0, b \geq 0, z \in \mathbb{R}^n\}$ and $\mathbb{M}_{n+1, \epsilon} := \{\mu \in \mathbb{M}_{n+1} \mid a \in [\frac{1}{4}, 1], 0 < b \leq \epsilon, z \in \mathbb{R}^n\}$.

Let $X := e^{\frac{1}{3}|y|^2} L^\infty(\mathbb{R}^n)$ with the corresponding norm. Using the implicit function theorem we will prove that for any $\mu_0 := (a_0, b_0, z_0) \in \mathbb{M}_{n+1, \epsilon_0}^{++}$ there exists a unique C^1 function $\tilde{g} : X \rightarrow \mathbb{M}_{n+1}$, defined in a neighborhood $\tilde{U}_{\mu_0} \subset X$ of V_{μ_0} , such that $G(\tilde{g}(u), u) = 0$ for all $u \in \tilde{U}_{\mu_0}$. Let $B_\epsilon(V_{\mu_0})$ and $B_\delta(\mu_0)$ be the balls in X and \mathbb{R}^{n+1} around V_{μ_0} and μ_0 and of the radii ϵ and δ , respectively.

Note first that the mapping G is C^1 and $G(\mu_0, V_{\mu_0}) = 0$ for all μ_0 . We claim that the linear map $\partial_\mu G(\mu_0, V_{\mu_0})$ is invertible.

Lemma 4. $\exists \epsilon_0 > 0$ such that $\partial_\mu G(\mu, u)$, for $u \in \tilde{U}_{\epsilon_0}$, is invertible.

Proof. Let the indices α and β run over the pairs (i, j) , $0 \leq i \leq j \leq n$. We compute

$$\partial_\mu G(\mu, u) = A_1 + A_2 \quad (28)$$

where the (α, β) -th entries of A_1 and A_2 are

$$A_1(\alpha, \beta) = \left\langle \partial_{\mu_\alpha} V_\mu, \varphi_{az}^{(\beta)} \right\rangle \quad (29)$$

and

$$A_2(\alpha, \beta) = \left\langle V_\mu - u, \partial_{\mu_\alpha} \varphi_{az}^{(\beta)} \right\rangle,$$

respectively. We write A_1 in the block form

$$A_1 = \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix},$$

where $K_{11} = \langle \partial_{(a, b^{\text{diag}})} V_\mu, \varphi_{az}^{ii} \rangle$, with $0 \leq i \leq n$, $K_{22} = \langle \partial_{b^{\text{off-diag}}} V_\mu, \varphi_{az}^{ij} \rangle$, with $1 \leq i < j \leq n$, $K_{33} = \langle \partial_z V_\mu, \varphi_{az}^{0i} \rangle$, with $1 \leq i \leq n$ and similarly for the other entries. For $b > 0$ and small, we compute using

change of variable $y = \lambda(x - z) - \alpha$, that (see Appendix 2 for more details)

$$K_{11} = \frac{\lambda^{-n+\frac{2}{p-1}}}{p-1} \left(\frac{a+\frac{1}{2}}{p-1} \right)^{\frac{1}{p-1}} \left(\frac{2\pi}{a} \right)^{\frac{n}{2}} \begin{pmatrix} \frac{1}{a+\frac{1}{2}} & \frac{1}{a+\frac{1}{2}} & \frac{1}{a+\frac{1}{2}} & \cdots & \frac{1}{a+\frac{1}{2}} \\ -\frac{1}{(p-1)a} & -\frac{3}{(p-1)a} & -\frac{1}{(p-1)a} & \cdots & -\frac{1}{(p-1)a} \\ -\frac{1}{(p-1)a} & -\frac{1}{(p-1)a} & -\frac{3}{(p-1)a} & \cdots & -\frac{1}{(p-1)a} \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ -\frac{1}{(p-1)a} & -\frac{1}{(p-1)a} & \cdots & -\frac{1}{(p-1)a} & -\frac{3}{(p-1)a} \end{pmatrix} + O(\|b\|) \quad (30)$$

is an $(n+1) \times (n+1)$ matrix,

$$K_{22} = -\lambda^{-n+\frac{2}{p-1}} \left(\frac{a+1/2}{p-1} \right)^{\frac{1}{p-1}} \frac{2}{(p-1)^2 a} \left(\frac{2\pi}{a} \right)^{n/2} I_{\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}} + O(\|b\|) \quad (31)$$

and

$$K_{33} = -\lambda^{-n+\frac{2}{p-1}} \left(\frac{a+1/2}{p-1} \right)^{\frac{1}{p-1}} \left(\frac{2\pi}{a} \right)^{n/2} b + o(\|b\|) \quad (32)$$

is an $n \times n$ matrix. Moreover,

$$K_{ij} = o(\|b\|) \text{ for } 1 \leq i \neq j \leq 3. \quad (33)$$

Since K_{11} , K_{22} and K_{33} are invertible, the matrix A_1 is also invertible. Furthermore, by the Schwarz inequality

$$\|A_2\| \lesssim \|u - V_{a_0 b_0}\|_X = O(\|b\|^2). \quad (34)$$

Therefore there exist ε_0 and ε_1 such that the matrix $\partial_\mu G(\mu, u)$ has an inverse for $\mu \in \mathbb{M}_{n+1, \varepsilon_0}$ and $u \in B_{\varepsilon_1}(V_\mu)$. \square

Moreover, from (28)-(34) we know that $\partial_\mu G$ can be written as

$$\partial_\mu G = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} + R,$$

where $A_{11} = O(1)$ and has an $O(1)$ inverse, $A_{22} = O(\|b\|)$ and has an $O(\|b\|^{-1})$ inverse, $A_{12} = o(\|b\|)$ and $A_{21} = o(\|b\|)$. Then we have

$$(\partial_\mu G)^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad (35)$$

where $B_{11} = (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} = O(1)$, $B_{22} = (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1} = O(\|b\|^{-1})$, $B_{12} = -A_{11}^{-1} A_{12} (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1} = o(1)$ and $B_{21} = -A_{22}^{-1} A_{21} (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} = o(1)$.

Hence by the implicit function theorem, the equation $G(\mu, u) = 0$ has a unique solution $\mu = \tilde{g}(u)$ on a neighborhood of every V_μ , $\mu \in \mathbb{M}_{n+1, \varepsilon}$, which is C^1 in u . Our next goal is to determine these neighborhoods.

To determine a domain of the function $\mu = \tilde{g}(u)$, we examine closely a proof of the implicit function theorem. Proceeding in a standard way, we expand the function $G(\mu, u)$ in μ around μ_0 :

$$G(\mu, u) = G(\mu_0, u) + \partial_\mu G(\mu_0, u)(\mu - \mu_0) + R(\mu, u),$$

where $R(\mu, u) = O(\|\mu - \mu_0\|^2)$ uniformly in $u \in X$. Here $|\mu|^2 = |a|^2 + \|b\|^2 + |z|^2$ for $\mu = (a, b, z)$. Inserting this into the equation $G(\mu, u) = 0$ and inverting the matrix $\partial_\mu G(\mu_0, u)$, we arrive at the fixed point problem

$\alpha = \Phi_u(\alpha)$, where $\alpha := \mu - \mu_0$ and $\Phi_u(\alpha) := -\partial_\mu G(\mu_0, u)^{-1}[G(\mu_0, u) + R(\mu, u)]$. By the above estimates there exists an ε_1 such that the matrix $\partial_\mu G(\mu_0, u)^{-1}$ is bounded in $u \in B_{\varepsilon_1}(V_{\mu_0})$. Define

$$|\mu|_{b_0} = |a| + \|b\| + \|b_0\|z$$

for $\mu = (a, b, z)$, then from (35) we have $|(\partial_\mu G)^{-1}\mu|_{b_0} \lesssim |\mu|$. It follows that

$$|\Phi_u(\alpha)|_{b_0} \lesssim |G(\mu_0, u)| + |\alpha|^2. \quad (36)$$

Furthermore, using that $\partial_\alpha \Phi_u(\alpha) = -\partial_\mu G(\mu_0, u)^{-1}[G(\mu, u) - G(\mu_0, u) + R(\mu, u)]$, we obtain that there exist $\varepsilon \leq \varepsilon_1$ and δ such that $\|\partial_\alpha \Phi_u(\alpha)\| \leq \frac{1}{2}$ for all $u \in B_\varepsilon(V_{\mu_0})$ and $\alpha \in B_\delta(0)$. Pick ε and δ so that $\varepsilon \ll \delta \ll \|b_0\| \ll 1$. Then, for all $u \in B_\varepsilon(V_{\mu_0})$, Φ_u is a contraction on the ball $B_\delta(0)$ and consequently has a unique fixed point in this ball. This gives a C^1 function $\mu = \tilde{g}(u)$ on $B_\varepsilon(V_{\mu_0})$ satisfying $|\mu - \mu_0| \leq \delta$. An important point here is that since $\varepsilon \ll \|b_0\|$ we have that $b > 0$ for all $V_{ab} \in B_\varepsilon(V_{\mu_0})$. Now, clearly, the balls $B_\varepsilon(V_{\mu_0})$ with $\mu_0 \in \mathbb{M}_{n+1, \varepsilon_0}$ cover the neighbourhood $\tilde{U}_{\varepsilon_0}$. Hence, the map \tilde{g} is defined on $\tilde{U}_{\varepsilon_0}$ and is unique, and the same is true for the map g , defined as $g(u_{\lambda, z_0}) = \tilde{g}(u)$, which implies the first part of the proposition.

Now we prove the second part of the proposition. The definition of the function $G(\mu, u)$ implies $G(\mu_0, u) = \lambda^{-n+\frac{2}{p-1}} \langle V_{a_0 b_0} - u_{\lambda, z_0}, \phi_a^{ij}(y) \rangle$, therefore

$$|G(\mu_0, u)| \lesssim \|e^{-\frac{1}{3}y^2}(u_{\lambda, z_0} - V_{a_0 b_0})\|_\infty. \quad (37)$$

This inequality together with the estimate (36) and the fixed point equation $\alpha = \Phi_u(\alpha)$, where $\alpha = \mu - \mu_0$ and $\mu = g(u_{\lambda, z_0})$, implies

$$|g(u_{\lambda, z_0}) - \mu_0|_{b_0} \lesssim \|e^{-\frac{1}{3}y^2}(u_{\lambda, z_0} - V_{a_0 b_0})\|_\infty. \quad (38)$$

From one of the conditions of the proposition, r.h.s. of (38) = $O(\|b_0\|^2)$ if $a_0 \in [\frac{1}{4}, 1]$. The last estimate implies (24) and (25). Using Equation (38) we obtain

$$\begin{aligned} \|\langle y \rangle^{-3}(u_{\lambda, z_0} - V_{g(u_{\lambda, z_0})})\|_\infty &\leq \|\langle y \rangle^{-3}(u_{\lambda, z_0} - V_{\mu_0})\|_\infty + \|\langle y \rangle^{-3}(V_{g(u_{\lambda, z_0})} - V_{\mu_0})\|_\infty \\ &\lesssim \|\langle y \rangle^{-3}(u_{\lambda, z_0} - V_{\mu_0})\|_\infty + |g(u_{\lambda, z_0}) - \mu_0| \\ &\lesssim \|\langle y \rangle^{-3}(u_{\lambda, z_0} - V_{\mu_0})\|_\infty, \end{aligned}$$

which leads to (26). Finally, to prove Equation (27), we write

$$\|u_{\lambda, z_0} - V_{g(u_{\lambda, z_0})}\|_\infty \leq \|u_{\lambda, z_0} - V_{a_0 b_0}\|_\infty + \|V_{g(u_{\lambda, z_0})} - V_{a_0 b_0}\|_\infty.$$

A straightforward computation gives $\|V_{ab} - V_{a_0 b_0}\|_\infty \lesssim |a - a_0| + \frac{\|b - b_0\|}{\|b_0\|}$. Since by (24), $|a - a_0| + \|b - b_0\| = O(\|b_0\|^2)$, we have $\|V_{ab} - V_{a_0 b_0}\|_\infty \lesssim \|b_0\|$. This together with the fact $\|u_{\lambda, z_0} - V_{a_0 b_0}\|_\infty \leq \delta_0$ completes the proof of (27). \square

Now we establish a reparametrization of the solution $u(x, t)$ on small time intervals. In Section 5 we convert this result to a global reparametrization. In the rest of the section it is convenient to work with the original time t , instead of rescaled time τ . We let $I_{t_0, \delta} := [t_0, t_0 + \delta]$ and define for any time t_0 and constant $\delta > 0$ three sets:

$$\mathcal{A}_{t_0, \delta} := C^1(I_{t_0, \delta}, [1/4, 1]), \quad \mathcal{B}_{t_0, \delta, \varepsilon_0} := C^1(I_{t_0, \delta}, \mathbb{M}_{n, \varepsilon_0}^+) \quad \text{and} \quad \mathcal{C}_{t_0, \delta} := C^1(I_{t_0, \delta}, [-1, 1]^n),$$

where we recall the constant ϵ_0 from Proposition 3.

Recall $u_{\lambda,z}(y, t) := \lambda(t)^{-\frac{2}{p-1}} u(x, t)$, with $x = z(t) + \lambda^{-1}(t)(y + \alpha(t))$. Suppose $u(\cdot, t)$ is a function such that for some $\lambda_0 > 0$

$$\sup_{t \in I_{t_0, \delta}} \|b^{-1}(t) \|\langle y \rangle^{-3} (u_{\lambda,z}(\cdot, t) - V_{a(t), b(t)})\|_{\infty} \ll 1 \quad (39)$$

for some $a \in \mathcal{A}_{t_0, \delta}$, $b \in \mathcal{B}_{t_0, \delta, \epsilon_0}$, $z \in \mathcal{C}_{t_0, \delta}$, $\lambda(t)$ satisfying $\lambda(t_0) = \lambda_0$ and $\lambda^{-3}(t) \partial_t \lambda(t) = a(t)$ and $\alpha(t)$ satisfying $\alpha(t_0) = \alpha_0$ and $\partial_t \alpha(t) - \lambda^2(t) a(t) \alpha(t) + \lambda(t) \partial_t z(t) = 0$. We define the set

$$\mathcal{U}_{t_0, \delta, \epsilon_0, \lambda_0, \alpha_0} := \{u \in C^1(\overset{\circ}{I}_{t_0, \delta}, \langle y \rangle^3 L^{\infty}(\mathbb{R}^n)) \mid (39) \text{ holds for some } a \in \mathcal{A}_{t_0, \delta}, b \in \mathcal{B}_{t_0, \delta, \epsilon_0} \text{ and } z \in \mathcal{C}_{t_0, \delta}\}.$$

Proposition 5. *Suppose $u \in \mathcal{U}_{t_0, \delta, \epsilon_0, \lambda_0, \alpha_0}$ and $\lambda_0^2 \delta \ll 1$. Then there exists a unique C^1 map $g_{\#} : \mathcal{U}_{t_0, \delta, \epsilon_0, \lambda_0, \alpha_0} \rightarrow \mathcal{A}_{t_0, \delta} \times \mathcal{B}_{t_0, \delta, \epsilon_0} \times \mathcal{C}_{t_0, \delta}$, such that for $t \in I_{t_0, \delta}$, $u(\cdot, t)$ can be uniquely represented in the form*

$$u_{\lambda}(y, t) = V_{g_{\#}(u)(t)}(y) + \xi(y, t), \quad (40)$$

with $(a(t), b(t), z(t)) = g_{\#}(u)(t)$ and

$$\begin{aligned} \xi(\cdot, t) &\perp \phi_{a(t)}^{(ij)} \text{ in } L^2(\mathbb{R}^n, e^{-\frac{a(t)}{2}|y|^2} dy), \\ \lambda^{-3}(t) \partial_t \lambda(t) &= a(t) \text{ and } \lambda(t_0) = \lambda_0, \\ \partial_t \alpha(t) - \lambda^2(t) a(t) \alpha(t) + \lambda(t) \partial_t z(t) &= 0 \text{ and } \alpha(t_0) = \alpha_0. \end{aligned} \quad (41)$$

Proof. For any function $a \in \mathcal{A}_{t_0, \delta}$, we define a function

$$\lambda(a, t) := (\lambda_0^{-2} - 2 \int_{t_0}^t a(s) ds)^{-\frac{1}{2}}.$$

Let $\lambda(a)(t) := \lambda(a, t)$. Next we define a function

$$\alpha(a, z)(t) := e^{\int_{t_0}^t \lambda^2(s) a(s) ds} \alpha_0 - \int_{t_0}^t e^{\int_s^t \lambda^2(\gamma) a(\gamma) d\gamma} \lambda(s) \partial_t z(s) ds.$$

Define the C^1 map $G_{\#}$:

$$C^1(I_{t_0, \delta}, \mathbb{R}^+) \times C^1(I_{t_0, \delta}, \mathbb{M}_n^+) \times C^1(I_{t_0, \delta}, \mathbb{R}^n) \times C^1(I_{t_0, \delta}, \langle y \rangle^3 L^{\infty}(\mathbb{R}^n)) \rightarrow C^1(I_{t_0, \delta}, \mathbb{R}^{\frac{(n+2)(n+1)}{2}})$$

as

$$G_{\#}(\mu, u)(t) := G(\mu(t), u_{\lambda(a), z}(\cdot, t)),$$

where $t \in I_{t_0, \delta}$, $\mu = (a, b, z)$ and $G(\mu, u)$ is the same as in the proof of Proposition 3. The orthogonality conditions on the fluctuation can be written as $G_{\#}(\mu, u) = 0$. Using the implicit function theorem we will first prove that for any $\mu_0 := (a_0, b_0, z_0) \in \mathcal{A}_{t_0, \delta} \times \mathcal{B}_{t_0, \delta, \epsilon_0} \times \mathcal{C}_{t_0, \delta}$ there exists a neighborhood \mathcal{U}_{μ_0} of V_{μ_0} and a unique C^1 map $g_{\#} : \mathcal{U}_{\mu_0} \rightarrow \mathcal{A}_{t_0, \delta} \times \mathcal{B}_{t_0, \delta, \epsilon_0} \times \mathcal{C}_{t_0, \delta}$ such that $G_{\#}(g_{\#}(v), v) = 0$ for all $v \in \mathcal{U}_{\mu_0}$.

We claim that $\partial_{\mu} G_{\#}(\mu, u)$ is invertible, provided $u_{\lambda(a), z}$ is close to V_{μ} . We compute

$$\partial_{\mu} G_{\#}(\mu, u)(t) = \partial_{\mu} G(\mu(t), u_{\lambda(a), z}(\cdot, t)) = A(t) + B(t), \quad (42)$$

where

$$A(t) := \partial_\mu G(\mu, v)|_{v=u_{\lambda(a),z}}, \quad B(t) := \partial_v G(\mu, v)|_{v=u_{\lambda(a),z}} \partial_\mu u_{\lambda(a),z}. \quad (43)$$

Note that in (43) $\partial_v G(\mu, v)|_{v=u_{\lambda(a),z}}$ is acting on $\partial_\mu u_{\lambda(a),z}$ as an integral with respect to y and let $B(t)(y)$ be the integral kernel of this operator. We have shown in Lemma 4 that the first term on the r.h.s. is invertible, provided $u_{\lambda(a),z}$ is close to V_μ .

Now we show that for $\delta > 0$ sufficiently small the second term on the r.h.s. is small. Let $v := u_{\lambda(a),z}$. Assuming for the moment that v is differentiable, we compute $\partial_a v = -\partial_a(\lambda^{-1})[\frac{2}{p-1}\lambda v - (y + \alpha)\nabla_y v] + \lambda^{-1}\partial_a \alpha \nabla_y v$. Combining the last two equations together with Equation (43) we obtain

$$[B(t)\rho](t) = \int B(t)(y)[(-\frac{2}{p-1}\lambda v + (y + \alpha)\nabla_y v)(\partial_a \lambda^{-1})\rho + \lambda^{-1}\nabla_y v(\partial_a \alpha)\rho]dy.$$

Integrating by parts in the second term in the parenthesis gives

$$[B(t)\rho](t) = - \int [(\frac{2}{p-1}\lambda v + v\nabla_y \cdot (y + \alpha))(\partial_a \lambda^{-1})\rho + \lambda^{-1}v\nabla_y \cdot (\partial_a \alpha)\rho]B(t)[y]dy. \quad (44)$$

Furthermore, $\partial_a(\lambda^{-1})\rho = \lambda(t) \int_{t_0}^t \rho(s)ds$ and

$$\begin{aligned} (\partial_a \alpha)\rho &= e^{\int_{t_0}^t \lambda^2(s)a(s)ds} \alpha_0 \int_{t_0}^t [a(s)\partial_a \lambda^2(s) + \lambda^2(s)]\rho(s)ds \\ &\quad - \int_{t_0}^t e^{\int_s^t \lambda^2(\gamma)a(\gamma)d\gamma} \partial_t z(s) [\lambda(s) \int_s^t (a(\gamma)\partial_a \lambda^2(\gamma) + \lambda^2(\gamma))\rho(\gamma)d\gamma + \partial_a \lambda(s)\rho(s)]ds. \end{aligned}$$

Now, using a density argument, we remove the assumption of the differentiability on v and conclude that (44) holds without this assumption. Using this expression and the inequality $\lambda(t) \leq \sqrt{2}\lambda_0$, provided $\delta \leq (4 \sup \alpha)^{-1}\lambda_0^{-2} \leq 1/4\lambda_0^{-2}$, we estimate

$$\|B(t)\rho\|_{L^\infty([t_0, t_0+\delta])} \lesssim \delta \lambda_0^2 \|v\|_{L^\infty} \|\rho\|_{L^\infty([t_0, t_0+\delta])}. \quad (45)$$

So $B(t)$ is small, if $\delta \lesssim (\lambda_0^2 \|v\|_{L^\infty})^{-1}$, as claimed. This shows that $\partial_\mu G_\#(\mu, u)$ is invertible, provided $u_{\lambda(a),z}$ is close to V_μ . Proceeding as in the proof of Proposition 3 we conclude the proof of Proposition 5. \square

4 A priori Estimates

Let $u(x, t)$, $0 \leq t \leq T$ be a solution to (1) with initial condition $u_0 \in U_{\epsilon_0}$ and $v(y, \tau) = \lambda^{-\frac{2}{p-1}}(t)u(x, t)$, where $y = \lambda(x - z) - \alpha$ and $\tau(t) := \int_0^t \lambda^2(s)ds$. We assume that there exist C^1 functions $a(\tau)$, $b(\tau)$, and $c(\tau)$ such that $v(y, \tau)$ can be represented as

$$v(y, \tau) = V_{a(\tau)b(\tau)} + \xi(y, \tau), \quad (46)$$

where, recall,

$$V_{ab} := \left(\frac{a + \frac{1}{2}}{p-1 + yby} \right)^{\frac{1}{p-1}}$$

where $\xi(\cdot, \tau) \perp \phi_{a(\tau)}^{(ij)}$ (see (23)), $\lambda^{-3}(t)\partial_t \lambda(t) = a(\tau(t))$. Since $u_0 \in U_{\epsilon_0}$, by condition (4)

$$\|\langle y \rangle^{-3} \xi(y, 0)\|_{\infty} \lesssim \|b(0)\|^2. \quad (47)$$

In this section we formulate a priori bounds on the fluctuation ξ which are proved in later sections.

Let the function $\tilde{\beta}(\tau)$ and the constant κ be defined as

$$\tilde{\beta}(\tau) := (b(0)^{-1} + \frac{4p\tau}{(p-1)^2}I)^{-1} \text{ and } \kappa := \min\{\frac{1}{2}, \frac{p-1}{2}\}, \quad (48)$$

and let $\beta(\tau)$ be the largest eigenvalue of $\tilde{\beta}(\tau)$. For the functions $\xi(\tau)$, $b(\tau)$ and $a(\tau)$ we introduce the following estimating functions (families of semi-norms)

$$\begin{aligned} M_1(T) &:= \max_{\tau \leq T} \beta^{-2}(\tau) \|\langle y \rangle^{-3} \xi(\tau)\|_{\infty}, \\ M_2(T) &:= \max_{\tau \leq T} \|\xi(\tau)\|_{\infty}, \\ A(T) &:= \max_{\tau \leq T} \beta^{-2}(\tau) \left| a(\tau) - \frac{1}{2} + \frac{2\text{Tr } b(\tau)}{p-1} \right|, \\ B(T) &:= \max_{\tau \leq T} \beta^{-(1+\kappa)}(\tau) \|b(\tau) - \tilde{\beta}(\tau)\|. \end{aligned} \quad (49)$$

Proposition 6. *Let ξ be defined in (46) and assume $M_1(0), A(0), B(0) \lesssim 1$, $M_2(0) \ll 1$. Assume there exists an interval $[0, T]$ such that for $\tau \in [0, T]$*

$$M(\tau), A(\tau), B(\tau) \leq \beta^{-\kappa/2}(\tau).$$

Then in the same time interval the parameters a , b and the function ξ satisfy the following estimates

$$\left| \frac{\partial}{\partial \tau} b(\tau) + \frac{4p}{(p-1)^2} b^2(\tau) \right| \lesssim \beta^3(\tau) + \beta^3(\tau) M_1(\tau) (1 + A(\tau)) + \beta^4(\tau) M_1^2(\tau) + \beta^{2p} M_1^{2p}(\tau), \quad (50)$$

and

$$B(\tau) \lesssim 1 + M_1(\tau) (1 + A(\tau)) + M_1^2(\tau) + M_1^p(\tau), \quad (51)$$

$$A(\tau) \lesssim A(0) + 1 + \beta(0) M_1(\tau) (1 + A(\tau)) + \beta(0) M_1^2(\tau) + \beta^{2p-2}(0) M_1^p(\tau), \quad (52)$$

$$\begin{aligned} M_1(\tau) &\lesssim M_1(0) + \beta^{\frac{\kappa}{2}}(0) [1 + M_1(\tau) A(\tau) + M_1^2(\tau) + M_1^p(\tau)] \\ &\quad + [M_2(\tau) M_1(\tau) + M_1(\tau) M_2^{p-1}(\tau)], \end{aligned} \quad (53)$$

$$\begin{aligned} M_2(\tau) &\lesssim M_2(0) + \beta^{1/2}(0) M_1(0) + \beta^{\frac{1}{3}}(0) M_1^{\frac{2}{3}}(T) M_2^{\frac{1}{3}}(T) + M_2^2(\tau) + M_2^p(\tau) \\ &\quad + \beta^{\frac{\kappa}{2}}(0) [1 + M_2(\tau) + M_1(\tau) A(\tau) + M_1^2(\tau) + M_1^p(\tau)]. \end{aligned} \quad (54)$$

Equations (50)-(52), (53) and (54) will be proved in Sections 7, 10 and 11 respectively.

Corollary 7. *Let ξ be defined in (46) and assume $M_1(0), A(0), B(0) \lesssim 1$, $M_2(0) \ll 1$. Assume there exists an interval $[0, T]$ such that for $\tau \in [0, T]$,*

$$M_1(\tau), A(\tau), B(\tau) \leq \beta^{-\kappa/2}(0).$$

Then in the same time interval the parameters a , b and the function ξ satisfy the following estimates

$$M_1(\tau), A(\tau), B(\tau) \lesssim 1, M_2(\tau) \ll 1. \quad (55)$$

(In fact, $M_i(\tau) \lesssim M_i(0) + \beta^{\frac{\kappa}{2}}(0)$, $i = 1, 2$.)

Proof. Since $\beta(\tau) \leq \beta(0) \ll 1$, we have

$$M_1(\tau), B(\tau), A(\tau) \leq \beta^{-\frac{\kappa}{2}}(0) \leq \beta^{-\frac{\kappa}{2}}(\tau), \quad (56)$$

where, recall, the definitions of $\beta(\tau)$ and κ are given in (48). Thus the conditions of the proposition above are satisfied. Since $M_1(\tau) \leq \beta^{-\frac{\kappa}{2}}(0)$, we can solve (52) for $A(\tau)$. We substitute the result into Equations (53) - (54) to obtain inequalities involving only the estimating functions $M_1(\tau)$ and $M_2(\tau)$. Consider the resulting inequality for $M_2(\tau)$. The only terms on the r.h.s., which do not contain $\beta(0)$ to a power at least $\kappa/2$ as a factor, are $M_2^2(\tau)$ and $M_2^p(\tau)$. Hence for $M_2(0) \ll 1$ this inequality implies that $M_2(\tau) \lesssim M_2(0) + \beta^{\frac{\kappa}{2}}(0)$. Substituting this result into the inequality for $M_1(\tau)$ we obtain that $M_1(\tau) \lesssim M_1(0) + \beta^{\frac{\kappa}{2}}(0)$ as well. The last two inequalities together with (51) and (52) imply the desired estimates on $A(\tau)$ and $B(\tau)$. \square

5 Proof of Main Theorem 1

We start with an auxiliary statement which eases the induction step. Recall the notation $I_{t_0, \delta} := [t_0, t_0 + \delta]$. We say that $\lambda(t)$ is *admissible* on $I_{t_0, \delta}$ if $\lambda \in C^1(I_{t_0, \delta}, \mathbb{R}^+)$ and $\lambda^{-3} \partial_t \lambda \in [1/4, 1]$. Recall that t_* is the maximal existence time defined in Section 1.

Lemma 8. *Assume $u \in C^1((0, t_*), \langle x \rangle^3 L^\infty)$, $t_0 \in [0, t_*)$ and $u_{\lambda_0}(\cdot, t_0) \in U_{\epsilon_0/2}$ for some λ_0 and for ϵ_0 given in Proposition 3. Then there are $\delta = \delta(\lambda_0, u) > 0$ and $\lambda(t)$, admissible on $I_{t_0, \delta}$, s.t. (40) and (41) hold on $I_{t_0, \delta}$.*

Proof. The conditions $u \in C^1((0, t_*), \langle x \rangle^3 L^\infty)$ and $u_{\lambda_0}(t_0) \in U_{\epsilon_0/2}$ imply that there is a $\delta = \delta(\lambda_0, u)$ s.t. $u \in \mathcal{U}_{t_0, \delta, \epsilon_0, \lambda_0, \alpha_0}$. By Proposition 5, the latter inclusion implies that there is $\lambda(t)$, admissible on $I_{t_0, \delta}$, $\lambda(t_0) = \lambda_0$, s.t. (40) and (41) hold on $I_{t_0, \delta}$. \square

Choose b_0 so that $C\|b_0\|^2 \leq \frac{1}{2}\epsilon_0$ with C the same as in (4) and with ϵ_0 given in Proposition 3. Let $v_0(y) := \lambda_0^{-\frac{2}{p-1}} u_0(z_0 + \lambda_0^{-1} y)$. Then $v_0 \in U_{\frac{1}{2}\epsilon_0}$, by condition (4) with $m = 3$, on the initial conditions. Hence Proposition 3 holds for v_0 and we have the splitting (23). Denote $g(v_0) =: (a(0), b(0), z(0))$.

Furthermore, by Lemma 8 there are $\delta_1 > 0$ and $\lambda_1(t)$, admissible on $[0, \delta_1]$, s.t. $\lambda_1(0) = \lambda_0$ and Equations (40) and (41) hold on the interval $[0, \delta_1]$. Hence, in particular, the estimating functions $M_1(\tau)$, $M_2(\tau)$, $A(\tau)$ and $B(\tau)$ of Section 5 are defined on the interval $[0, \delta_1]$. We will write these functions in the original time t , i.e. we will write $M_i(t)$ for $M_i(\tau(t))$ where $\tau(t) = \int_0^t \lambda^2(s) ds$.

Recall the definitions of $\beta(\tau)$ and κ given in (48). Since $\beta(0)$ is the largest eigenvalue of $b(0)$, by Equation (4) and Proposition 3, $A(0)$, $M_1(0) \lesssim 1$ and $M_2(0) \ll 1$, while $B(0) \ll 1$, by definition. We have, by continuity, that

$$M_1(t), A(t), B(t) \leq \beta^{-\frac{\kappa}{2}}(0), \quad (57)$$

for a sufficiently small time interval, which we can take to be $[0, \delta_1]$. Then by Corollary 7 we have that for the same time interval

$$M_1(t), A(t), B(t) \lesssim 1, M_2(t) \ll 1. \quad (58)$$

Equation (58) implies that $u_{\lambda_1}(\cdot, \delta_1) \in U_{\epsilon_0/2}$ (indeed, by the definitions of $M_1(t)$ and $M_2(t)$ we have $\|\langle y \rangle^{-3}(u_{\lambda_1}(\cdot, t) - V_{a(t), b(t)})\| \leq M_1(t)|b(t)|^2$ and $\|u(t)\|_\infty \lesssim \lambda_1^{-\frac{2}{p-1}}(t)[1 + M_1(t) + M_2(t)]$). Now we can apply

Lemma 8 again and find $\delta_2 > 0$ and $\lambda_2(t)$, admissible on $[0, \delta_1 + \delta_2]$, s.t. $\lambda_2(t) = \lambda_1(t)$ for $t \in [0, \delta_1]$ and Equations (40) and (41) hold on the interval $[0, \delta_1 + \delta_2]$.

We iterate the procedure above to show that there is a maximal time $t^* \leq t_*$ (t_* is the maximal existence time), and a function $\lambda(t)$, admissible on $[0, t^*)$, s.t. (40) and (41) and (58) hold on $[0, t^*)$. We claim that $t^* = t_*$ and $t^* < \infty$ and $\lambda(t^*) = \infty$. Indeed, if $t^* < t_*$ and $\lambda(t^*) < \infty$, then by the a priori estimate (58) $u_\lambda(t) \in U_{\epsilon_0/2}$ for any $t \leq t^*$. By Lemma 8, this implies that there is $\delta > 0$ and $\lambda_\#(t)$, admissible on $[0, t^* + \delta]$, s.t. (40) and (41) hold on $[0, t^* + \delta]$ and $\lambda_\#(t) = \lambda(t)$ on $[0, t^*)$, which would contradict the assumption that the time t^* is maximal. Hence

$$\text{either } t^* = t_* \text{ or } t^* < t_* \text{ and } \lambda(t^*) = \infty. \quad (59)$$

The second case in (59) is ruled out as follows. Using the relation between the functions $u(x, t)$ and $v(y, \tau)$ we obtain the following a priori estimate on the (non-rescaled) solution $u(x, t)$ of equation (1):

$$\|u(t)\|_\infty \lesssim \lambda(t)^{\frac{2}{p-1}} [1 + M_1(t) + M_2(t)], \quad (60)$$

where we used the fact $\|\xi(\cdot, \tau(t))\|_\infty \lesssim M_1(t) + M_2(t)$. By the estimate (58) above the majorants $M_j(t)$ are uniformly bounded and therefore

$$\|u(t)\|_\infty \lesssim \lambda(t)^{\frac{2}{p-1}} \text{ for } t < t^*. \quad (61)$$

Moreover (46) and the fact $\|y\|^{-3}\xi\|_\infty \lesssim \|b(t)\|^2$, implied by $M_1 \lesssim 1$, give

$$|u(0, t)| \geq \lambda(t)^{\frac{2}{p-1}} \left[\left(\frac{c(t)}{p-1} \right)^{\frac{1}{p-1}} - C\|b(t)\|^2 \right] \rightarrow \infty, \quad (62)$$

as $t \uparrow t^*$, which implies that $t^* \geq t_*$ and therefore $t_* = t^*$.

Now we consider the first case in (59). In this case we must have either $t^* = t_* = \infty$ or $t^* = t_* < \infty$ and $\lambda(t^*) = \infty$, since otherwise we would have existence of the solution on an interval greater than $[0, t_*)$. Finally, the case $t^* = t_* = \infty$ is ruled out in the next paragraph. This proves the claim which can reformulated as: there is a function $\lambda(t)$, admissible on $[0, t_*)$, s.t. (40) and (41) and (58) hold on $[0, t_*)$ and $\lambda(t) \rightarrow \infty$ as $t \rightarrow t_*$. This gives the statements (1) and (2) of Theorem 1.

By the definitions of $A(t)$ and $B(t)$ in (49) and the facts that $A(t), B(t) \lesssim 1$ proved above, we have that

$$a(t) - \frac{1}{2} = -\frac{2}{p-1} \text{Tr } b(\tau) + O(\beta^2(\tau)), \quad b(t) = \tilde{\beta}(\tau) + O(\beta^{1+\kappa/2}(\tau)), \quad (63)$$

where, recall, $\tau = \tau(t) = \int_0^t \lambda^2(s) ds$. Hence $a(t) - \frac{1}{2} = O(\beta(\tau))$. Recall that $a = \lambda^{-3} \partial_t \lambda$, which can be rewritten as $\lambda^{-2}(t) = \lambda_0^{-2} - 2 \int_0^t a(s) ds$ or $\lambda(t) = [\lambda_0^{-2} - 2 \int_0^t a(s) ds]^{-\frac{1}{2}}$. Assume $t^* = \infty$. Since $|a(t) - \frac{1}{2}| = O(\beta(\tau))$, there exists a time $t^{**} < \infty$ such that $\lambda_0^{-2} = 2 \int_0^{t^{**}} a(s) ds$, i.e. $\lambda(t) \rightarrow \infty$ as $t \rightarrow t^{**}$. This contradicts the assumption that $\lambda(t)$ is defined on $[0, t^* = \infty)$. Hence $t^* < \infty$. This completes the proof of Statements (1) and (2) of Theorem 1.

Now we prove statement (3) of Theorem 1. Equation (63) implies $b(t) \rightarrow 0$ and $a(t) \rightarrow \frac{1}{2}$ as $t \rightarrow t^*$. By the analysis above and the definitions of a , τ and $\tilde{\beta}$ (see (48)) we have

$$\lambda(t) = (t^* - t)^{-\frac{1}{2}}(1 + o(1)), \quad \tau(t) = -\ln|t^* - t|(1 + o(1)),$$

and

$$\tilde{\beta}(\tau(t)) = -\frac{(p-1)^2}{4p \ln |t^* - t|} (I + o(1)).$$

This gives the first equation in (6). By (63) and the relation $c = a + \frac{1}{2}$ we have the second and third equations in (6). Finally, let $\zeta(t) = z(t) + \alpha(t)/\lambda(t)$. By (18) and (25) we obtain the last equation in (6). This completes the proof of Theorem 1.

□

6 Lyapunov-Schmidt Splitting (Effective Equations)

According to Lemma 8, the solution $v(y, \tau)$ of (20) can be decomposed as (46), with the parameters a and b , and the fluctuation ξ depending on time τ :

$$v = V_{ab} + \xi, \quad \xi \perp \phi_a^{(ij)}, \quad 0 \leq i, j \leq n, \quad (64)$$

in the sense of $L^2(\mathbb{R}^n, e^{-a|y|^2/2} dy)$, where $V_{ab} := \left(\frac{c}{p-1+aby} \right)^{\frac{1}{p-1}}$, $c = a + \frac{1}{2}$ and $\phi_a^{(ij)}$ are defined in the beginning of Section 3. Plugging the decomposition (64) into (20) gives the equation (see Appendix 2 for details)

$$\xi_\tau = -\mathcal{L}_{ab}\xi + \mathcal{N}(\xi, a, b) + \mathcal{F}(a, b) \quad (65)$$

where

$$\mathcal{L}_{ab} = -\Delta_y + ay \cdot \nabla_y + \frac{2a}{p-1} - \frac{pc}{p-1+aby}, \quad (66)$$

$$\mathcal{N}(\xi, a, b) = |\xi + V_{ab}|^{p-1}(\xi + V_{ab}) - V_{ab}^p - pV_{ab}^{p-1}\xi \quad (67)$$

$$\mathcal{F}(a, b) = \frac{1}{p-1} \left[\Gamma_0 + \sum_{j,k} \Gamma_{jk} \frac{(p-1)ay_j y_k}{p-1+aby} + G_1 \right] V_{ab}, \quad (68)$$

with the functions Γ_{jk} ($1 \leq j, k \leq n$) given as

$$\Gamma_0 := -\frac{c_\tau}{c} + (c-2a) - \frac{2}{p-1} \text{Tr } b, \quad (69)$$

$$\Gamma_{jk} := \frac{1}{a(p-1)} \left(\partial_\tau b_{jk} - (c-2a)b_{jk} + \frac{2b_{jk}}{p-1} \text{Tr } b + \frac{4p}{(p-1)^2} \sum_{i=1}^n b_{ij} b_{ik} \right), \quad (70)$$

$$G_1 := -\frac{4p(aby)(\sum_{i=1}^n (\sum_{j=1}^n b_{ij} y_j)^2)}{(p-1)^2(p-1+aby)^2}. \quad (71)$$

Proposition 9. *If $A(\tau)$, $B(\tau) \leq \beta^{-\frac{\kappa}{2}}(\tau)$ and $1/4 \leq c(0) \leq 1$, then*

$$\|\langle y \rangle^{-3} \mathcal{F}\|_\infty = O \left(|\Gamma_0| + \sum_{j,k} |\Gamma_{jk}| + \beta^{\frac{5}{2}} \right) \quad (72)$$

and

$$\|\mathcal{F}\|_\infty = O\left(|\Gamma_0| + \frac{1}{\beta} \sum_{j,k} |\Gamma_{jk}| + \beta\right). \quad (73)$$

where, recall, $\langle y \rangle := (1 + y_1^2 + \dots + y_n^2)^{\frac{1}{2}}$. Furthermore we have for $\mathcal{N} = \mathcal{N}(\xi, a, b)$

$$|\mathcal{N}| \lesssim |\xi|^2 + |\xi|^p. \quad (74)$$

Proof. We estimate $\|\langle y \rangle^{-3} \mathcal{F}\|_\infty$ using the expression of \mathcal{F} and the estimates

$$\|V_{ab}\|_\infty, \|\langle y \rangle^{-3} y_j y_k\|_\infty \lesssim 1.$$

The result is

$$\|\langle y \rangle^{-3} \mathcal{F}\|_\infty \lesssim |\Gamma_0| + \sum_{j,k} |\Gamma_{jk}| + \|b\|^{\frac{5}{2}}. \quad (75)$$

Now we estimate $\|\mathcal{F}\|$. Recall the expression of \mathcal{F} in Equation (68). We use the estimates

$$\|V_{ab}\|_\infty, \left\| \frac{b_{jk} y_j y_k}{(p-1+by)^2} V_{ab} \right\|_\infty \lesssim 1$$

to obtain that

$$\|\mathcal{F}\|_\infty \lesssim |\Gamma_0| + \sum_{j,k} \frac{1}{\|b\|} |\Gamma_{jk}| + \|b\|. \quad (76)$$

To complete the proof we estimate b in terms of β and B of the first bound. The assumption that $B \leq \beta^{-\frac{\kappa}{2}}$ implies that $\|b\| = \beta + O(\beta^{1+\frac{\kappa}{2}})$, which together with estimates (75) and (76), implies estimates (72) and (73).

For (74) we observe that if $V_{ab} \leq 2|\xi|$ then $|\mathcal{N}| \leq (3^p + 2^p + p2^{p-1})|\xi|^p$. If $V_{ab} \geq 2|\xi|$, then we use the formula $\mathcal{N} = p \int_0^1 \left[(V_{ab} + s\xi)^{p-1} - V_{bc}^{p-1} \right] \xi ds$ and consider the cases $1 < p \leq 2$ and $p > 2$ separately to obtain (74). \square

Recall that $\phi_a^{(ij)} = (\sqrt{a}y_i)^{1-\delta_{i0}}(\sqrt{a}y_j)^{1-\delta_{j0}}$, $i, j = 0, \dots, n$.

Proposition 10. Suppose that $A(\tau), M_1(\tau) \leq \beta^{-\frac{\kappa}{2}}$, $B(\tau) \leq \beta^{-\frac{\kappa}{2}}$ and $1/2 \leq c(0) \leq 2$ for $0 \leq \tau \leq T$. Let $v = V_{ab} + \xi$ be a solution to (20) with $\xi \perp \phi_a^{(ij)}$ in $L^2(\mathbb{R}^n, e^{-a|y|^2/2} dy)$. Over times $0 \leq \tau \leq T$, the parameters a and b satisfy the equations

$$\partial_\tau b = -\frac{4p}{(p-1)^2} b^2 - \frac{2b}{p-1} \text{Tr } b + (c-2a)b + \mathcal{R}_b(\xi, a, b), \quad (77)$$

$$\frac{\partial_\tau a}{a + \frac{1}{2}} = \left(\frac{1}{2} - a\right) - \frac{2}{p-1} \text{Tr } b + \mathcal{R}_a(\xi, a, b), \quad (78)$$

where the remainders \mathcal{R}_a and \mathcal{R}_b are of the order $O(\beta^3 + \beta^3 M_1(1+A) + \beta^4 M_1^2 + \beta^{2p} M_1^p)$ and satisfy $\mathcal{R}_b(0, a, b), \mathcal{R}_c(0, a, b) = O(\beta^3)$.

Proof. We take the inner product of equation (65) with $\phi_a^{(ij)}$ to get

$$\langle \xi_\tau, \phi_a^{(ij)} \rangle = \langle -\mathcal{L}_{ab}\xi + \mathcal{N}(\xi, a, b) + \mathcal{F}(a, b), \phi_a^{(ij)} \rangle.$$

We start with analyzing the \mathcal{F} term. The inner product of \mathcal{F} with $\phi_a^{(ij)}$ gives the expressions

$$\begin{aligned} (p-1) \langle \mathcal{F}, \phi_a^{(ij)} \rangle &= \Gamma_0 \langle V_{ab}, \phi_a^{(ij)} \rangle + \langle G_1 V_{ab}, \phi_a^{(ij)} \rangle \\ &\quad + \sum_{k,l} \left[\Gamma_{kl} \langle V_{ab}, ay_k y_l \phi_a^{(ij)} \rangle - \Gamma_{kl} \langle \frac{ay_k y_l y b y}{p-1+yby} V_{ab}, \phi_a^{(ij)} \rangle \right]. \end{aligned} \quad (79)$$

By rescaling the variable of integration so that the exponential term does not contain the parameter a and expanding V_{ab} in b we obtain the estimates

$$\begin{aligned} \langle V_{ab}, \phi_a^{(ij)} \rangle &= \left(\frac{a + \frac{1}{2}}{p-1} \right)^{\frac{1}{p-1}} \left(\frac{2\pi}{a} \right)^{\frac{n}{2}} \delta_{ij} + O(\|b\|), \\ \langle V_{ab}, ay_k y_l \phi_a^{(ij)} \rangle &= (\delta_{kl} \delta_{ij} (1 + 2\delta_{ik}) + (1 - \delta_{kl})(\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il})) \left(\frac{2\pi}{a} \right)^{\frac{n}{2}} + O(\|b\|), \\ \left\langle \frac{ay_k y_l y b y}{p-1+yby} V_{ab}, \phi_a^{(ij)} \right\rangle &= O(\|b\|), \\ \langle G_1 V_{ab}, \phi_a^{(ij)} \rangle &= O(\|b\|^3), \end{aligned}$$

Where we recall $G_1 := -\frac{4p(yby)(\sum_{i=1}^n (\sum_{j=1}^n b_{ij} y_j)^2)}{(p-1)^2(p-1+yby)^2}$. Substituting these estimates into Equation (79) gives

$$(p-1) \langle \mathcal{F}, \phi_a^{(00)} \rangle = \left(\frac{a + \frac{1}{2}}{p-1} \right)^{\frac{1}{p-1}} \left(\frac{2\pi}{a} \right)^{\frac{n}{2}} (\Gamma_0 + \sum_k \Gamma_{kk}) + R_1, \quad (80)$$

$$(p-1) \langle \mathcal{F}, \phi_a^{(ii)} \rangle = \left(\frac{a + \frac{1}{2}}{p-1} \right)^{\frac{1}{p-1}} \left(\frac{2\pi}{a} \right)^{\frac{n}{2}} (\Gamma_0 + \sum_k \Gamma_{kk} + 2\Gamma_{ii}) + R_2 \text{ for } 1 \leq i \leq n, \quad (81)$$

$$(p-1) \langle \mathcal{F}, \phi_a^{(ij)} \rangle = 2 \left(\frac{a + \frac{1}{2}}{p-1} \right)^{\frac{1}{p-1}} \left(\frac{2\pi}{a} \right)^{\frac{n}{2}} \Gamma_{ij} + R_3 \text{ for } 1 \leq i < j \leq n, \quad (82)$$

where the remainders R_1 , R_2 and R_3 are bounded by $O(\|b\|(|\Gamma_0| + \sum_{i,j} |\Gamma_{ij}|) + \|b\|^3)$.

To estimate the remaining terms we differentiate the orthogonality condition $\langle \xi, \phi_a^{ij} \rangle = 0$ to obtain

$$0 = \langle \xi_\tau, \phi_a^{ij} \rangle + \langle \xi, \partial_\tau \phi_a^{ij} \rangle - \frac{a_\tau}{2} \langle \xi, \phi_a^{ij} |y|^2 \rangle,$$

where the last term is due to the weight $e^{-\frac{a}{2}|y|^2}$. Now compute

$$\langle \xi, \partial_\tau \phi_a^{ij} \rangle = 0 \text{ and } \left| \frac{a_\tau}{2} \langle \xi, \phi_a^{ij} |y|^2 \rangle \right| \leq \left| \frac{1}{2} a^{-1} a_\tau \langle \langle y \rangle^{-3} \xi, a^2 \langle y \rangle^3 y_i y_j |y|^2 \rangle \right|.$$

Estimating the right hand side of the second inequality by Hölder's inequality and using the definition of $M_1(\tau)$ gives that over times $0 \leq \tau \leq T$

$$\left| \frac{a_\tau}{2} \langle \xi, \phi_a^{ij} |y|^2 \rangle \right| = O(|a_\tau| \beta^2 M_1).$$

Next we estimate a_τ . Since $c = \frac{1}{2} + a$, we have $a_\tau = c_\tau$, and so we find from (69)

$$c_\tau = O(\Gamma_0 + \beta^2 A)$$

for times $0 \leq \tau \leq T$, and hence

$$\left| \frac{a_\tau}{2} \langle \xi, \phi_a^{(ij)} | y|^2 \rangle \right| = O(\beta^2 M_1 (|\Gamma_0| + \beta^2 A)). \quad (83)$$

We now estimate the terms involving the linear operator \mathcal{L}_{abc} . Write the operator \mathcal{L}_{abc} as

$$\mathcal{L}_{ab} = \mathcal{L}_* - \frac{pc}{p-1+yby},$$

where $\mathcal{L}_* := -\Delta_y + ay \cdot \nabla_y + \frac{2a}{p-1}$ is self-adjoint on $L^2(\mathbb{R}^n, e^{-\frac{a}{2}|y|^2})$ and satisfies $\mathcal{L}_* \phi_a^{(00)} = \frac{2a}{p-1} \phi_a^{(00)}$ and $\mathcal{L}_* \phi_a^{(ij)} = \frac{2ap}{p-1} \phi_a^{(ij)} + 2\delta_{ij}$ for $1 \leq i, j \leq n$. Projecting $\mathcal{L}_{abc} \xi$ onto the eigenvectors $\phi_a^{(00)}$ and $\phi_a^{(ij)}$ of \mathcal{L}_* gives the equations

$$\begin{aligned} \langle \mathcal{L}_{ab} \xi, \phi_a^{(00)} \rangle &= - \left\langle \xi, \frac{pc}{p-1+yby} \right\rangle = \frac{pc}{p-1} \left\langle \xi, \frac{yby}{p-1+yby} \right\rangle, \\ \langle \mathcal{L}_{ab} \xi, \phi_a^{(ij)} \rangle &= - \left\langle \xi, \frac{pcay_i y_j}{p-1+yby} \right\rangle = \frac{pc}{p-1} \left\langle \xi, \frac{ay_i y_j yby}{p-1+yby} \right\rangle. \end{aligned}$$

Estimating with Hölder's inequality gives the inequalities

$$\begin{aligned} |\langle \mathcal{L}_{ab} \xi, \phi_a^{(00)} \rangle| &\lesssim \|b\| \|\langle y \rangle^{-3} \xi\|_\infty \\ |\langle \mathcal{L}_{ab} \xi, \phi_a^{(ij)} \rangle| &\lesssim \|b\| \|\langle y \rangle^{-3} \xi\|_\infty. \end{aligned}$$

In terms of the estimating functions β and M_1 , these estimates, after using the above estimate of a_τ and simplifying in a and c , become

$$\langle \mathcal{L}_{ab} \xi, \phi_a^{(00)} \rangle \lesssim \beta^3 M_1 \quad (84)$$

$$\langle \mathcal{L}_{ab} \xi, \phi_a^{(ij)} \rangle \lesssim \beta^3 M_1. \quad (85)$$

Lastly, we estimate the inner products involving the nonlinearity. Because of (74), both $\langle \mathcal{N}, \phi_a^{(00)} \rangle$ and $\langle \mathcal{N}, \phi_a^{(ij)} \rangle$ are estimated by $O(\|\langle y \rangle^{-3} \xi\|_\infty^2 + \|\langle y \rangle^{-3} \xi\|_\infty^p)$. Writing this in terms of β and M_1 and simplifying gives the estimate

$$|\langle \mathcal{N}, \phi_a^{(00)} \rangle|, |\langle \mathcal{N}, \phi_a^{(ij)} \rangle| \lesssim \beta^4 M_1^2 + \beta^{2p} M_1^p. \quad (86)$$

Estimates (80)-(86) imply that Γ_0 and Γ_{ij} are of the order

$$O\left(\beta(|\Gamma_0| + \sum_{i,j} |\Gamma_{ij}|) + \beta^3 + \beta^2 M_1 (\beta + |\Gamma_0| + \beta^2 A) + \beta^4 M_1^2 + \beta^{2p} M_1^p\right).$$

By the facts that $\beta(\tau) \leq \beta_0 \ll 1$ and $A, M_1 \leq \beta^{-\frac{\kappa}{2}}$, we obtain the estimates

$$|\Gamma_0| + \sum_{i,j} |\Gamma_{ij}| \lesssim \beta^3 + \beta^3 M_1(1+A) + \beta^4 M_1^2 + \beta^{2p} M_1^p \quad (87)$$

for the times $0 \leq \tau \leq T$. □

Equations (72), (73) and (87) yield the following corollary.

Corollary 11. *Let $k_0 := \min\{1, 2p-1\}$ and $k_3 := \min\{5/2, 2p\}$. Then for $m = 0$ and 3*

$$\|\langle y \rangle^{-m} \mathcal{F}\|_\infty \lesssim \beta^{k_m}(\tau)[1 + M_1(1+A) + M_1^2 + M_1^p]. \quad (88)$$

7 Proof of Estimates (50)-(52)

Recall that $a = c - \frac{1}{2}$. Assume $B(\tau) \leq \beta^{-\frac{\kappa}{2}}(\tau)$ for $\tau \in [0, T]$ which implies that $\tilde{\beta} \lesssim b \lesssim \tilde{\beta}$. We rewrite equation (77) as $\partial_\tau b = -\frac{4p}{(p-1)^2}b^2 + b\left(\frac{1}{2} - a - \frac{2\text{Tr } b}{p-1}\right) + \mathcal{R}_b$. By the definition of A , the second term on the right hand side is bounded by $\|b\|\beta^2 A \lesssim \beta^3 A$. Thus, using the bound for \mathcal{R}_b given in Proposition 10, we obtain (50).

To prove (51) we use the inequality $\beta I \lesssim b$ to obtain the estimate

$$\left\| -\partial_\tau b^{-1} + \frac{4p}{(p-1)^2} I \right\| \lesssim \beta + \beta M_1(1+A) + \beta^2 M_1^2 + \beta^{2p-2} M_1^p. \quad (89)$$

Since $\tilde{\beta}$ is a solution to $-\partial_\tau \tilde{\beta}^{-1} + 4p(p-1)^{-2} I = 0$, Equation (89) implies that

$$\left\| \partial_\tau (b^{-1} - \tilde{\beta}^{-1}) \right\| \lesssim \beta + \beta M_1(1+A) + \beta^2 M_1^2 + \beta^{2p-2} M_1^p.$$

Integrating this equation over $[0, \tau]$, multiplying the result by $\beta^{-1-\kappa}$ and using that $\tilde{\beta}(0) = b(0)$, gives the estimate

$$\beta^{-1-\kappa} \|\tilde{\beta} - b\| \lesssim \beta^{1-\kappa} \int_0^\tau (\beta + \beta M_1(1+A) + \beta^2 M_1^2 + \beta^{2p-2} M_1^p) ds,$$

where, recall, $\kappa := \min\{\frac{1}{2}, \frac{p-1}{2}\} < 1$. Hence, by the definition of β and B and the facts that M_1 and A are increasing functions, (51) follows.

Define the quantity $\Gamma := \frac{1}{2} - a - \frac{2}{p-1} \text{Tr } b$. Differentiating Γ with respect to τ and substituting for $\partial_\tau b$ and $a_\tau = c_\tau$. From equations (77) and (78), we obtain

$$\partial_\tau \Gamma = -c(\Gamma + \mathcal{R}_c) - \frac{2}{p-1} \text{Tr} \left(-\frac{4p}{(p-1)^2} b^2 - \frac{2b}{p-1} \text{Tr } b + (c-2a)b + \mathcal{R}_b \right).$$

Replacing $b(c-2a)$ by $b\Gamma + \frac{2b}{p-1} \text{Tr } b$ and rearranging the resulting equation gives that

$$\partial_\tau \Gamma + \left[a + \frac{1}{2} + \frac{2}{p-1} \text{Tr } b \right] \Gamma = \frac{8p}{(p-1)^3} \text{Tr } b^2 - (a + \frac{1}{2}) \mathcal{R}_c - \frac{2}{p-1} \mathcal{R}_b.$$

Let $\mu = \exp \left(\int_0^\tau \left(a + \frac{1}{2} + \frac{2}{p-1} \text{Tr } b \right) ds \right)$. We now integrate the above equation over $[0, \tau] \subseteq [0, T]$. Then the above equation implies that

$$\mu(\tau)\Gamma(\tau) - \mu(0)\Gamma(0) = \int_0^\tau \partial_\tau(\mu\Gamma) = \frac{8p}{(p-1)^3} \int_0^\tau \mu \text{Tr } b^2 ds - \int_0^\tau \left(a + \frac{1}{2} \right) \mu \mathcal{R}_c ds - \int_0^\tau \frac{2}{p-1} \mu \mathcal{R}_b ds.$$

Use the inequality $\|b\| \lesssim \beta$ and the estimates of \mathcal{R}_{b_i} and \mathcal{R}_c in Proposition 10 to obtain

$$|\Gamma| \lesssim \mu^{-1}\Gamma(0) + \mu^{-1} \int_0^\tau \mu \beta^2 ds + \mu^{-1} \int_0^\tau \mu \left(\beta^3 + \beta^3 M_1(1+A) + \beta^4 M_1^2 + \beta^{2p} M_1^p \right) ds.$$

For our purpose, it is sufficient to use the less sharp inequality

$$|\Gamma| \lesssim \mu^{-1}\Gamma(0) + (1 + \beta(0)M_1(1+A) + \beta^2(0)M_1^2 + \beta^{2p-2}(0)M_1^p) \mu^{-1} \int_0^\tau \mu \beta^2 ds.$$

The assumption that $A(\tau), B(\tau) \leq \beta^{-\frac{2}{p-1}}(\tau)$, implies that $a + \frac{1}{2} + \frac{2}{p-1} \text{Tr } b = 1 + O(\beta^2 A) \geq \frac{1}{2}$ and therefore $\beta^{-2}\mu^{-1} \lesssim \beta^{-2}(0)$ and $\int_0^\tau \mu(s)\beta^2(s) ds \lesssim \mu(\tau)\beta^2(\tau)$. The last two inequalities and the relation $\max_{s \leq \tau} \beta^{-2}(s)|\Gamma(s)| = A(\tau)$ lead to (52).

8 Rescaling of Fluctuations on a Fixed Time Interval

The coefficient in front of $|y|^2$ in the operator \mathcal{L}_{ab} , (66), is time dependent, complicating the estimation of the semigroup generated by this operator. In this section we introduce new time and space variables in such a way that the coefficient of $|y|^2$ in the new operator is constant (cf [3, 4, 6, 38]).

Let T be given and let $t(\tau)$ be the inverse of the function $\tau(t) := \int_0^t \lambda^2(s) ds$. We approximate the scaling parameter $\lambda(t)$ over the time interval $[0, t(T)]$ by a new parameter $\lambda_1(t)$. We choose $\lambda_1(t)$ to satisfy for $t \leq t(T)$

$$\partial_t (\lambda_1^{-3} \partial_t \lambda_1) = 0 \text{ with } \lambda_1(t(T)) = \lambda(t(T)) \text{ and } \partial_t \lambda_1(t(T)) = \partial_t \lambda(t(T)).$$

We define $\alpha := \lambda_1^{-3} \partial_t \lambda_1 = a(T)$. This is an analog of the parameter a and it is constant. The last two conditions imply that λ_1 is tangent to λ at $t = t(T)$. Define the new time and space variables as

$$z = \frac{\lambda_1}{\lambda} y \text{ and } \sigma = \sigma(t(\tau)) \text{ with } \sigma(t) := \int_0^t \lambda_1^2(s) ds$$

where $\tau \leq T$, $\sigma \leq S := \sigma(T)$ and λ, λ_1 are functions of $t(\tau)$. Now we introduce the new function $\eta(z, \sigma)$ by the equality

$$\lambda_1^{\frac{2}{p-1}} \eta(z, \sigma) = \lambda^{\frac{2}{p-1}} \xi(y, \tau). \quad (90)$$

Denote by $t(\sigma)$ the inverse of the function $\sigma(t)$. In the equation for $\eta(z, \sigma)$ derived below and in what follows the symbols λ, a and b stand for $\lambda(t(\sigma)), a(\tau(t(\sigma)))$ and $b(\tau(t(\sigma)))$, respectively. Substituting this change of variables into (65) gives the governing equation for η :

$$\partial_\sigma \eta = -L_\alpha \eta + W(a, b, \alpha) \eta + F(a, b, \alpha) + N(\eta, a, b, \alpha), \quad (91)$$

where

$$\begin{aligned} L_\alpha &:= L_0 + V, \quad L_0 := -\Delta_z + \alpha z \cdot \nabla_z - 2\alpha, \quad V := \frac{2p\alpha}{p-1} - \frac{2p\alpha}{p-1+z\tilde{\beta}z}, \\ W(a, b, \alpha) &:= \frac{\lambda^2}{\lambda_1^2} \frac{p(a + \frac{1}{2})}{p-1 + \frac{\lambda^2}{\lambda_1^2} z b z} - \frac{2p\alpha}{p-1+z\tilde{\beta}z}, \\ F(a, b, \alpha) &:= \left(\frac{\lambda}{\lambda_1} \right)^{\frac{2p}{p-1}} \mathcal{F}(a, b, c) \end{aligned} \quad (92)$$

and

$$N(\eta, a, b, \alpha) := \left(\frac{\lambda}{\lambda_1} \right)^{\frac{2p}{p-1}} \mathcal{N} \left(\left(\frac{\lambda_1}{\lambda} \right)^{\frac{2}{p-1}} \eta, b, c \right),$$

where, recall, c and a are related as $c = a + \frac{1}{2}$ and β is defined in (48).

In the next statement we prove that the new parameter $\lambda_1(t)$ is a good approximation of the old one, $\lambda(t)$. The proof is an exact copy of the one in [6]. We reproduce it for completeness. We have

Proposition 12. *If $A(\tau) \leq \beta^{-\frac{\kappa}{2}}(\tau)$ and $\beta(0) \ll 1$, then*

$$\left| \frac{\lambda}{\lambda_1}(t(\tau)) - 1 \right| \lesssim \beta(\tau) \leq \beta(0). \quad (93)$$

Proof. Differentiating $\frac{\lambda}{\lambda_1} - 1$ with respect to τ (recall that $\frac{dt}{d\tau} = \frac{1}{\lambda^2}$) gives the expression

$$\frac{d}{d\tau} \left(\frac{\lambda}{\lambda_1} - 1 \right) = \frac{\lambda}{\lambda_1} a - \frac{\lambda_1}{\lambda} \alpha$$

or, after some manipulations,

$$\frac{d}{d\tau} \left(\frac{\lambda}{\lambda_1} - 1 \right) = 2a \left(\frac{\lambda}{\lambda_1} - 1 \right) + \Gamma \quad (94)$$

with

$$\Gamma := a - \alpha - a \frac{\lambda_1}{\lambda} \left(\frac{\lambda}{\lambda_1} - 1 \right)^2 + (a - \alpha) \left(\frac{\lambda_1}{\lambda} - 1 \right).$$

Observe that $\frac{\lambda}{\lambda_1}(t(\tau)) - 1 = 0$ when $\tau = T$. Thus Equation (94) can be rewritten as

$$\frac{\lambda}{\lambda_1}(t(\tau)) - 1 = - \int_\tau^T e^{-\int_\tau^\sigma 2a(\rho) d\rho} \Gamma(\sigma) d\sigma. \quad (95)$$

By the definition of $A(\tau)$ and the definition $\alpha = a(T)$ we have that, if $A(\tau) \leq \beta^{-\frac{\kappa}{2}}(\tau)$, then

$$|a(\tau) - \alpha|, \quad \left| a(\tau) - \frac{1}{2} \right| \leq 2\beta(\tau) \quad (96)$$

on the time interval $\tau \in [0, T]$. Thus

$$|\Gamma| \lesssim \beta + \left(1 + \frac{\lambda_1}{\lambda} \right) \left(\frac{\lambda}{\lambda_1} - 1 \right)^2 + \beta \left| \frac{\lambda}{\lambda_1} - 1 \right|. \quad (97)$$

which together with (95) and (96) implies (93). \square

9 Estimate of the Propagators

Let P^α be the orthogonal projection onto the orthogonal complement of the space spanned by the eigenvectors of L_0 corresponding to the smallest three eigenvalues. Denote by $V_\alpha(\tau, \sigma)$ the propagator generated by the operator $-P^\alpha L_\alpha P^\alpha$ on $\text{Ran } P^\alpha$, where, recall, the operator L_α is defined in (92). The main result of this section is the following theorem.

Theorem 13. *For any function $g \in \text{Ran } P^\alpha$ and for $c_0 := \alpha - \epsilon$ with some $\epsilon > 0$ small we have*

$$\|\langle z \rangle^{-3} V_\alpha(\tau, \sigma) g\|_\infty \lesssim e^{-c_0(\tau - \sigma)} \|\langle z \rangle^{-3} g\|_\infty.$$

The proof of this theorem is given after Lemma 18. We observe that in the L^2 -norm $P^\alpha L_\alpha P^\alpha \geq (-\Delta_z + \alpha z \cdot \nabla_z - 2\alpha) P^\alpha \geq \frac{1}{2} \alpha P^\alpha$. However, this does not help in proving the weighted L^∞ bound above. Recall the definition of the operator $L_0 := -\Delta_z + \alpha z \cdot \nabla_z - 2\alpha$ in (92) and define $U_0(x, y)$ as the integral kernel of the operator e^{-rL_0} . We begin with

Lemma 14. *For $k = 0, 1, 2, 3, 4$, any function g and $r > 0$ we have that*

$$\|\langle z \rangle^{-k} e^{-L_0 r} g\|_\infty \lesssim e^{2\alpha r} \|\langle z \rangle^{-k} g\|_\infty \quad (98)$$

or equivalently

$$\int \langle x \rangle^{-k} U_0(x, y) \langle y \rangle^k dy \lesssim e^{2\alpha r}. \quad (99)$$

Proof. We only prove the case $k = 2$. The cases $k = 0, 4$ are similar. The cases $k = 1, 3$ follow from $k = 0, 2, 4$ by an interpolation result. For the case $k = 2$, using that the integral kernel of e^{-rL_0} is positive and therefore $\|e^{-rL_0} g\|_\infty \leq \|f^{-1} g\|_\infty \|e^{-rL_0} f\|_\infty$ for any $f > 0$ and using that $e^{-rL_0} 1 = e^{2\alpha r} 1$ and $e^{-rL_0}(\alpha|z|^2 - n) = (\alpha|z|^2 - n)$, we find that

$$\begin{aligned} \|\langle z \rangle^{-2} e^{-rL_0} g\|_\infty &\leq \|\langle z \rangle^{-2} e^{-rL_0} (|z|^2 + 1)\|_\infty \|\langle z \rangle^{-2} g\|_\infty \\ &= \|\langle z \rangle^{-2} [e^{2\alpha r} (\frac{n}{\alpha} + 1) + (|z|^2 - \frac{n}{\alpha})]\|_\infty \|\langle z \rangle^{-2} g\|_\infty \\ &\leq 2(\frac{n}{\alpha} + 1) e^{2\alpha r} \|\langle z \rangle^{-2} g\|_\infty. \end{aligned}$$

This implies (98). To prove (99) we use that $U_0(x, y)$ is, by definition, the integral kernel of the operator e^{-rL_0} , and take $g(x) = \langle x \rangle^k$ in (98) to obtain (99). \square

Next we prove a more refined bound on the free evolution e^{-rL_0} .

Lemma 15. *For any function g and positive constant r we have*

$$\|\langle z \rangle^{-3} e^{-rL_0} P^\alpha g\|_\infty \lesssim e^{-\alpha r} \|\langle z \rangle^{-3} g\|_\infty.$$

Proof. First, we decompose the projection P^α in a convenient way. We write the operator L_0 as

$$L_0 = \sum_{k=1}^n L_0^{(k)} - 2\alpha, \quad \text{where} \quad L_0^{(k)} := -\partial_{z_k}^2 + \alpha z_k \partial_{z_k}. \quad (100)$$

The spectra of the operators $L_0^{(k)}$ are:

$$\sigma\left(L_0^{(k)}\right) = \{m\alpha \mid m = 0, 1, 2, \dots\}. \quad (101)$$

Let $P_0^{(k)}$, $P_1^{(k)}$ and $P_2^{(k)}$ be the orthogonal projections onto the eigenspaces of the operator $L_0^{(k)}$ corresponding to the first, the second and third eigenvalues of $L_0^{(k)}$, respectively, and let

$$\begin{aligned} P_3^{(k)} &:= 1 - P_0^{(k)} - P_1^{(k)} - P_2^{(k)}, \\ P_{0'}^{(k)} &:= 1, \quad P_{1'}^{(k)} := 1 - P_0^{(k)}, \\ P_{2'}^{(k)} &:= 1 - P_0^{(k)} - P_1^{(k)}. \end{aligned}$$

Then for any k , we have

$$\begin{aligned} P_0^{(k)} + P_1^{(k)} + P_2^{(k)} + P_3^{(k)} &= 1, \\ P_{0'}^{(k)} &= 1, \quad P_0^{(k)} + P_{1'}^{(k)} = 1, \\ P_0^{(k)} + P_1^{(k)} + P_{2'}^{(k)} &= 1. \end{aligned} \quad (102)$$

Let $\vec{i} = (i_1, i_2, \dots, i_n)$, $i_j = 0, 0', 1, 1', 2, 2', 3$, $|\vec{i}| = \sum_{j=1}^n i_j$, where the primed numbers are counted as the usual ones, and $P_{\vec{i}} = P_{i_1}^{(1)} P_{i_2}^{(2)} \dots P_{i_n}^{(n)}$. For every $k \in \{1, \dots, n\}$, we introduce the set

$$I_k \equiv I_k^{(n)} = \{\vec{i} = (i_1, \dots, i_n) \mid \text{either } |\vec{i}| = 3 \text{ and } i_k \neq 0', 1', 2', \text{ or } |\vec{i}| < 3 \text{ and } i_j \neq 0', 1', 2' \forall 1 \leq j \leq n\}.$$

Then we have the following lemma, whose proof is given in Appendix 2:

Lemma 16. *For any $1 \leq k \leq n$, there exists a subset J_k of I_k such that $1 = \sum_{\vec{i} \in J_k} P_{\vec{i}}$.*

Since for any k

$$\sum_{\vec{i} \in I_k, |\vec{i}|=j, i_l \neq 0', 1', 2' \forall l} P_{\vec{i}}$$

is the eigenprojection corresponding to the j -th eigenvalue of L_0 , $j = 0, 1, 2$, we have, by the definition of P^α and Lemma 16, that

$$P^\alpha = \sum_{|\vec{i}|=3, \vec{i} \in J_k} P_{\vec{i}}, \quad \forall k \in \{1, 2, \dots, n\}. \quad (103)$$

Equations (100) and (103) give

$$e^{-rL_0} P^\alpha = \sum_{|\vec{i}|=3, \vec{i} \in J_k} e^{-rL_0} P_{\vec{i}} = e^{2\alpha r} \sum_{|\vec{i}|=3, \vec{i} \in J_k} \prod_{j=1}^n \left(e^{-rL_0^{(j)}} P_{i_j}^{(j)} \right). \quad (104)$$

In the following, it is convenient to use the notation $z_0 := 1$. By the inequality $\langle z \rangle^3 \lesssim \sum_{k=0}^n |z_k|^3$, we have

$$\left\| \langle z \rangle^{-3} e^{-rL_0} P^\alpha \langle z \rangle^3 \right\|_{L^\infty \rightarrow L^\infty} \lesssim \left\| \langle z \rangle^{-3} e^{-rL_0} P^\alpha \sum_{k=0}^n |z_k|^3 \right\|_{L^\infty \rightarrow L^\infty} \lesssim \sum_{k=0}^n A_k, \quad (105)$$

where $A_k = \|\langle z \rangle^{-3} e^{-rL_0} P^\alpha |z_k|^3\|_{L^\infty \rightarrow L^\infty}$ for $0 \leq k \leq n$. Now by (104) and $\langle z \rangle^{-3} \leq \prod_{j=1}^n \langle z_j \rangle^{-i_j}$, we obtain that for $0 \leq k \leq n$,

$$\begin{aligned} A_k &\leq \sum_{\vec{i} \in J_k, |\vec{i}|=3} \|\langle z \rangle^{-3} e^{-rL_0} P_{\vec{i}} |z_k|^3\|_{L^\infty \rightarrow L^\infty} \\ &\lesssim \sum_{\vec{i} \in J_k, |\vec{i}|=3} \left\| \prod_{j=1}^n \langle z_j \rangle^{-i_j} e^{-rL_0} P_{\vec{i}} |z_k|^3 \right\|_{L^\infty \rightarrow L^\infty} \\ &= e^{2\alpha r} \sum_{\vec{i} \in J_k, |\vec{i}|=3} \left\| \prod_{j=1}^n \left(\langle z_j \rangle^{-i_j} e^{-rL_0^{(j)}} P_{i_j}^{(j)} |z_k|^{3\delta_{jk}} \right) \right\|_{L^\infty \rightarrow L^\infty}. \end{aligned}$$

We claim that, if $\vec{i} \in J_k$, then

$$\left\| \langle z_j \rangle^{-i_j} e^{-rL_0^{(j)}} P_{i_j}^{(j)} |z_k|^{3\delta_{jk}} \right\|_{L^\infty \rightarrow L^\infty} \lesssim e^{-i_j \alpha r}, \quad (106)$$

Indeed if $j = k \geq 1$, then $i_k \neq 0', 1', 2'$. For $i_j = 0, 1$, or 2 (106) follows from the relation $e^{-rL_0^{(j)}} P_{i_j}^{(j)} = e^{-i_j \alpha r} P_{i_j}^{(j)}$ which is due to the definition of $P_{i_j}^{(j)}$. For $i_j = 3$ it is proved in [6] using integration by parts (see Appendix 2). If $j \neq k$ (which is, in particular, the case when $k = 0$), the proof is similar. Then by the above two inequalities and the relation $\sum_{j=1}^n i_j = |\vec{i}| = 3$, we obtain

$$A_k \lesssim e^{-\alpha r} \text{ for } 0 \leq k \leq n. \quad (107)$$

Equations (105) and (107) imply the statement of the lemma. \square

Next, we estimate the propagator $U_\alpha(\tau, \sigma)$, generated by the operator $-L_\alpha$.

Proposition 17. *For any function g and positive constants σ and r we have*

$$\|\langle z \rangle^{-3} U_\alpha(\sigma + r, \sigma) P^\alpha g\|_\infty \lesssim [e^{2\alpha r} r(1+r) \beta^{1/2}(\sigma) + e^{-\alpha r}] \|\langle z \rangle^{-3} g\|_\infty. \quad (108)$$

Proof. Let $B_\lambda, \lambda \in \frac{R}{2}\mathbb{Z}^n$, be a collection of semi-open, disjoint boxes centered at λ , of sidelength R , whose union is \mathbb{R}^n . We take $R \leq \frac{1+r}{2}$. Let $g_\lambda(x) = g(x)\chi_\lambda(x)$, where $\chi_\lambda(x)$ is the characteristic function of B_λ . Then $g(x) = \sum_\lambda g_\lambda(x)$. Let

$$E(x, y) = \int_\sigma^{\sigma+r} e^{-\int_\sigma^{\sigma+r} V(\sigma+s, \omega(s) + \omega_0(s)) ds} d\mu(\omega), \quad (109)$$

where $d\mu(\omega)$ is an n -dimensional harmonic oscillator (Ornstein-Uhlenbeck) probability measure on the continuous paths $\omega : [\sigma, \sigma + r] \rightarrow \mathbb{R}$ with the boundary condition $\omega(\sigma) = \omega(\sigma + r) = 0$ and

$$\omega_0(s) = e^{\alpha(\tau-s)} \frac{e^{2\alpha\sigma} - e^{2\alpha s}}{e^{2\alpha\sigma} - e^{2\alpha\tau}} x + e^{\alpha(\sigma-s)} \frac{e^{2\alpha\tau} - e^{2\alpha s}}{e^{2\alpha\tau} - e^{2\alpha\sigma}} y. \quad (110)$$

It is shown in the Appendix that

$$|\partial_y E(x, y)| \lesssim r\beta^{\frac{1}{2}}. \quad (111)$$

Recall that $U_\alpha(\tau, \sigma)$ is the evolution generated by $-L_\alpha$. Let $U(x, y)$ and $U_0(x, y)$ be the integral kernels of the operators $U_\alpha(\sigma + r, \sigma)$ and e^{-rL_0} , respectively. By Feynmann-Kac formula (154), proved in the Appendix, we have that $U(x, y) = U_0(x, y)E(x, y)$. Then

$$U_\alpha(\sigma + r, \sigma)P^\alpha g(x) = \int U_0(x, y)E(x, y)P^\alpha g(y)dy \quad (112)$$

$$= \sum_\lambda \int U_0(x, y)E(x, y)P^\alpha g_\lambda(y)dy =: A(x) + B(x), \quad (113)$$

where

$$A(x) := \sum_\lambda \int U_0(x, y)E(x, \lambda)P^\alpha g_\lambda(y)dy$$

and

$$B(x) := \sum_\lambda \int U_0(x, y)[E(x, y) - E(x, \lambda)]P^\alpha g_\lambda(y)dy.$$

First we estimate the function A . We rewrite $A(x) = \int U_0(x, y)P^\alpha g_x(y)dy = (e^{-rL_0}P^\alpha g_x)(x)$ with $g_x(y) = \sum_\lambda E(x, \lambda)g_\lambda(y)$. Now by Lemma 15 we have

$$\begin{aligned} \|\langle x \rangle^{-3}A\|_\infty &= \|\langle x \rangle^{-3}e^{-rL_0}P^\alpha g_x\|_\infty \\ &\lesssim e^{-\alpha r} \sup_y |\langle y \rangle^{-3}g_x(y)|. \end{aligned}$$

Since $|E(x, \lambda)| \leq 1$ and g_λ 's have disjoint supports), we obtain $|g_x(y)| \leq \sum_\lambda |g_\lambda| = |g|$. The last two inequalities give

$$\|\langle x \rangle^{-3}A\|_\infty \lesssim e^{-\alpha r} \|\langle x \rangle^{-3}g\|_\infty. \quad (114)$$

Next we estimate the function B . Using $U_0(x, y) > 0$, (111), Mean Value Theorem and the fact that the diameters of B_λ are not greater than $1 + r$, we obtain

$$|B(x)| \lesssim r(1+r)\beta^{\frac{1}{2}} \int U_0(x, y) \sum_\lambda |P^\alpha g_\lambda(y)|dy = r(1+r)\beta^{\frac{1}{2}} \left(e^{-rL_0} \sum_\lambda |P^\alpha g_\lambda(y)| \right) (x). \quad (115)$$

Thus by (115), Lemma 14 and the relation $|g| = \sum_\lambda |g_\lambda|$,

$$\|\langle x \rangle^{-3}B\|_\infty \lesssim r(1+r)\beta^{\frac{1}{2}}e^{2\alpha r} \|\langle x \rangle^{-3}g\|_\infty. \quad (116)$$

Combining (113), (114) and (116), we obtain the estimate (108). This proves Proposition 17. \square

We will also need the following lemma

Lemma 18.

$$\|\langle z \rangle^{-k}U_\alpha(\tau, \sigma)g\|_\infty \leq e^{2\alpha(\tau-\sigma)}\|\langle z \rangle^{-k}g\|_\infty \quad (117)$$

with $k = 0$ or 3 .

Proof. By Equations (109) and (154) we have that $|U_\alpha(\tau, \sigma)|(x, y) \leq e^{-L_0(\tau-\sigma)}(x, y)$. Thus we have

$$\|\langle z \rangle^{-k} U_\alpha(\tau, \sigma) g\|_\infty \leq \|\langle z \rangle^{-k} e^{-L_0(\tau-\sigma)} |g|\|_\infty. \quad (118)$$

Now we use Lemma 14 to estimate the right hand side to complete the proof. \square

Proof of Theorem 13. Recall that \bar{P}_α is the projection on the span of the three first eigenfunctions of the operator L_0 and $P^\alpha := 1 - \bar{P}^\alpha$. We write

$$L_\alpha = P^\alpha L_\alpha P^\alpha + \bar{P}^\alpha L_\alpha \bar{P}^\alpha + E_1, \quad (119)$$

where the operator E_1 is defined as $E_1 := \bar{P}^\alpha L_\alpha P^\alpha + P^\alpha L_\alpha \bar{P}^\alpha$. Using that $\bar{P}^\alpha P^\alpha = 0$, we transform E_1 as

$$E_1 = -\bar{P}^\alpha \frac{2p\alpha z \tilde{\beta} z}{(p-1)(p-1+z\beta z)} P^\alpha - P^\alpha \frac{2p\alpha z \tilde{\beta} z}{(p-1)(p-1+z\beta z)} \bar{P}^\alpha.$$

This relation implies that

$$\|\langle z \rangle^{-3} E_1 \eta(\sigma)\|_\infty \lesssim \beta(\tau(\sigma)) \|\langle z \rangle^{-3} \eta(\sigma)\|_\infty. \quad (120)$$

We use the Duhamel principle to rewrite the propagator $V_\alpha(\sigma_1, \sigma_2)$ on $\text{Ran } P^\alpha$ as

$$V_\alpha(\sigma_1, \sigma_2) P^\alpha = U_\alpha(\sigma_1, \sigma_2) P^\alpha - \int_{\sigma_2}^{\sigma_1} U_\alpha(\sigma_1, s) E_1 V_\alpha(s, \sigma_2) P^\alpha ds. \quad (121)$$

Let $r = \sigma_1 - \sigma_2$, $g \in \text{Ran } P^\alpha$ and $\eta(\sigma_1) := V_\alpha(\sigma_1, \sigma_2)g$. We claim that if $e^{\alpha r} \leq \beta(\tau(\sigma_2))^{-1/8}$ then we have

$$\|\langle z \rangle^{-3} \eta(\sigma_1)\|_\infty \lesssim e^{-\alpha r} \|\langle z \rangle^{-3} \eta(\sigma_2)\|_\infty. \quad (122)$$

To prove the claim we compute each term on the right hand side of (121).

(A) Notice that $P^\alpha \eta(s) = \eta(s)$. We use Proposition 17 to obtain for $e^{\alpha r} \leq \beta(\tau(\sigma_2))^{-1/8}$ that

$$\|\langle z \rangle^{-3} U_\alpha(\sigma_1, \sigma_2) g\|_\infty \lesssim e^{-\alpha r} \|\langle z \rangle^{-3} g\|_\infty. \quad (123)$$

(B) By Lemma 18 and (120) we obtain

$$\|\langle z \rangle^{-3} \int_{\sigma_2}^{\sigma_1} U_\alpha(\sigma_1, s) E_1 \eta(s) ds\|_\infty \lesssim \int_{\sigma_2}^{\sigma_1} e^{2\alpha(\sigma_1-s)} \beta(\tau(s))^{\frac{1}{2}} \|\langle z \rangle^{-3} \eta(s)\| ds.$$

Using the condition $e^{\alpha r} \leq \beta(\sigma_2)^{-1/8}$ and the relation $\beta(\tau(s)) \leq \beta(\tau(\sigma_2))$ for $s \geq \sigma_2$ again, we find

$$\|\langle z \rangle^{-3} \int_{\sigma_2}^{\sigma_1} U_\alpha(\sigma_1, s) E_1 \eta(s) ds\|_\infty \lesssim \int_{\sigma_2}^{\sigma_1} e^{-\alpha(\sigma_1-s)} \beta(\tau(s))^{\frac{1}{2}} \|\langle z \rangle^{-3} \eta(s)\| ds. \quad (124)$$

Equations (121), (123) and (124) imply for $e^{\alpha r} \leq \beta^{-1/8}(\tau(\sigma_2))$ that (remember that $\eta(\sigma_2) = g$)

$$\|\langle z \rangle^{-3} \eta(\tau)\|_\infty \lesssim e^{-\alpha r} \|\langle z \rangle^{-3} \eta(\sigma_2)\|_\infty + \int_{\sigma_2}^{\tau} e^{-\alpha(\tau-s)} \beta(\tau(s))^{\frac{1}{2}} \|\langle z \rangle^{-3} \eta(s)\| ds. \quad (125)$$

Next, we define a function $K(r)$ as

$$K(r) := \max_{0 \leq s \leq r} e^{\alpha s} \|\langle z \rangle^{-3} \eta(\sigma_2 + s)\|. \quad (126)$$

Then (125) implies that

$$K(\sigma_1 - \sigma_2) \lesssim \|\langle z \rangle^{-3} \eta(\sigma_2)\|_\infty + \int_{\sigma_2}^{\sigma_1} \beta(\tau(s))^{\frac{1}{2}} ds K(\sigma_1 - \sigma_2).$$

We observe that

$$\int_{\sigma_2}^{\sigma_1} \beta(\tau(s))^{\frac{1}{2}} ds \leq 1/2$$

if $e^{\alpha r} \leq \beta(\tau(\sigma_2))^{-1/8}$ and if $\beta(0) \ll 1$ and, therefore, $\beta(\tau(s)) = \frac{1}{\frac{1}{\beta(0)} + \frac{4p}{(p-1)^2} \tau(s)}$ are small. Thus we have

$$K(\sigma_1 - \sigma_2) \lesssim \|\langle z \rangle^{-3} \eta(\sigma_2)\|_\infty,$$

which together with Equation (126) implies (122). Writing

$$V_\alpha(\tau, \sigma) = V_\alpha(\sigma_1, \sigma_2) V_\alpha(\sigma_2, \sigma_3) \cdots V_\alpha(\sigma_{m-1}, \sigma_m)$$

with $\sigma_1 = \tau$, $\sigma_m = \sigma$ and $|\sigma_i - \sigma_{i+1}| = r$ such that $e^{\alpha r} \leq \beta^{-1/8}(\tau(\sigma_k)) \forall k$ and iterating (122) completes the proof of the theorem. □

10 Estimate of $M_1(\tau)$ (Equation (53))

In this subsection we derive an estimate for $M_1(T)$ given in Equation (53). Given any time τ' , choose $T = \tau'$ and pass from the unknown $\xi(y, \tau)$, $\tau \leq T$, to the new unknown $\eta(z, \sigma)$, $\sigma \leq S$, given in (90). Now we estimate the latter function. To this end we use Equation (91). Observe that the function η is not orthogonal to the first three eigenvectors of the operator L_0 defined in (92). Thus we apply the projection P^α to Equation (91) to get

$$\frac{d}{d\sigma} P^\alpha \eta = -P^\alpha L_\alpha P^\alpha \eta + P^\alpha \sum_{k=1}^4 D_k, \quad (127)$$

where we used the fact that P^α are τ -independent and the functions $D_k \equiv D_k(\sigma)$, $k = 1, 2, 3, 4$, are defined as

$$\begin{aligned} D_1 &:= -P^\alpha V \eta + P^\alpha V P^\alpha \eta, & D_2 &:= W(a, b, \alpha) \eta, \\ D_3 &:= F(a, b, \alpha), & D_4 &:= N(\eta, a, b, \alpha), \end{aligned}$$

recall the definitions of the functions V , W , F and N after (92).

Lemma 19. *If $A(\tau), B_i(\tau) \leq \beta^{-\frac{\kappa}{2}}(\tau)$ for $\tau \leq T$ and $\|b_0\| \ll 1$, then we have*

$$\|\langle z \rangle^{-3} D_1(\sigma)\|_\infty \lesssim \beta^{5/2}(\tau(\sigma)) M_1(T), \quad (128)$$

$$\|\langle z \rangle^{-3} D_2(\sigma)\|_\infty \lesssim \beta^{2+\frac{\kappa}{2}}(\tau(\sigma)) M_1(T), \quad (129)$$

$$\|\langle z \rangle^{-3} D_3(\sigma)\|_\infty \lesssim \beta^{\min\{5/2, 2p\}}(\tau(\sigma)) [1 + M_1(T)(1 + A(T)) + M_1^2(T) + M_1^p(T)], \quad (130)$$

$$\begin{aligned} \|\langle z \rangle^{-3} D_4\|_\infty &\lesssim \beta^2(\tau(\sigma)) M_1(T) [\beta^{1/2}(\tau(\sigma)) M_1(T) + M_2(T) \\ &\quad + \beta^{\frac{p-1}{2}}(\tau(\sigma)) M_1^{p-1}(T) + M_2^{p-1}(T)]. \end{aligned} \quad (131)$$

Proof. In what follows we use the following estimates, implied by (93),

$$\frac{\lambda_1}{\lambda}(t(\tau)) - 1 = O(\beta(\tau)), \text{ thus } \frac{\lambda_1}{\lambda}(t(\tau)), \frac{\lambda}{\lambda_1}(t(\tau)) \leq 2, \langle z \rangle^{-3} \lesssim \langle y \rangle^{-3} \quad (132)$$

where, recall that $z := \frac{\lambda_1}{\lambda}y$. We start with proving the following two estimates which will be used frequently below

$$\|\eta(\sigma)\|_\infty \lesssim \beta^{1/2}(\tau(\sigma))M_1(\tau(\sigma)) + M_2(\tau(\sigma)) \leq \beta^{1/2}(\tau(\sigma))M_1(T) + M_2(T), \quad (133)$$

$$\|\langle z \rangle^{-3}\eta(\sigma)\|_\infty \lesssim \beta^2(\tau(\sigma))M_1(\tau(\sigma)) \leq \beta^2(\tau(\sigma))M_1(T). \quad (134)$$

Denote by $\chi_{\geq D}$ and $\chi_{\leq D}$ the characteristic functions of the sets $\{|x| \geq D\}$ and $\{|x| \leq D\}$:

$$\chi_{\geq D}(x) := \begin{cases} 1, & \text{if } |x| \geq D \\ 0, & \text{otherwise} \end{cases} \quad \text{and } \chi_{\leq D} := 1 - \chi_{\geq D}. \quad (135)$$

We take $D := \frac{C}{\sqrt{\beta}}$ where C is a large constant. Writing $1 = 1 - \chi_{\geq D} + \chi_{\geq D}$ and using the inequality $1 - \chi_{\geq D} \lesssim \beta^{-3/2}(\tau)\langle y \rangle^{-3}$, the relation between ξ and η , see (90), and Estimate (132) we find

$$\begin{aligned} \|\eta(\sigma)\|_\infty &\lesssim \|\xi(\tau(\sigma))\|_\infty \lesssim \beta^{-3/2}(\tau(\sigma))\|\langle y \rangle^{-3}\xi(\tau(\sigma))\|_\infty \\ &+ \|\chi_{\geq D}\xi(\tau)\|_\infty \leq \beta^{1/2}(\tau(\sigma))M_1(\tau(\sigma)) + M_2(\tau(\sigma)) \end{aligned} \quad (136)$$

which is (133). Similarly recall that $z = \frac{\lambda_1}{\lambda}y$ which together with (90) and (132) yields

$$\|\langle z \rangle^{-3}\eta(\sigma)\|_\infty \lesssim \|\langle y \rangle^{-3}\xi(\tau(\sigma))\|_\infty \lesssim \beta^2(\tau(\sigma))M_1(\tau(\sigma)) \leq \beta^2(\tau(\sigma))M_1(T).$$

Thus we have (134).

Now we proceed directly to proving the lemma. First we rewrite D_1 as

$$D_1(\sigma) = -P^\alpha \frac{2p\alpha z \tilde{\beta}(\tau(\sigma))z}{(p-1)(p-1+z\tilde{\beta}(\tau(\sigma))z)}(1-P^\alpha)\eta(\sigma).$$

Now, using that $\langle z \rangle^{-1} \frac{zbz}{p-1+zbz} \lesssim \|b\|^{1/2}$ and that $\|b\| \lesssim \beta$, we obtain

$$\|\langle z \rangle^{-3}D_1(\sigma)\|_\infty \lesssim \beta^{1/2}(\tau)\|\langle z \rangle^{-2}(1-P^\alpha)\eta(\sigma)\|_\infty.$$

Next, because of the explicit form of $\bar{P}^\alpha := 1 - P^\alpha$, i.e. $\bar{P}^\alpha = |\phi_{0,\alpha}\rangle\langle\phi_{0,\alpha}| + \sum_{i=1}^n |\phi_{1,\alpha}^{(i)}\rangle\langle\phi_{1,\alpha}^{(i)}| + \sum_{i=1}^n |\phi_{2,\alpha}^{(i)}\rangle\langle\phi_{2,\alpha}^{(i)}| +$

$\sum_{1 \leq i \neq j \leq n} |\phi_{2,\alpha}^{(ij)}\rangle\langle\phi_{2,\alpha}^{(ij)}|$, where $\phi_{m,\alpha}$ are the normalized eigenfunctions of the operator $L_0 := -\Delta_z + \alpha z \cdot \partial_z - 2\alpha$, we have for any function g

$$\|\langle z \rangle^{-2}\bar{P}^\alpha g\|_\infty \lesssim \|\langle z \rangle^{-3}g\|_\infty. \quad (137)$$

Collecting the estimates above and using (134), we arrive at

$$\|\langle z \rangle^{-3}D_1(\sigma)\|_\infty \lesssim \beta^{1/2}(\tau(\sigma))\|\langle z \rangle^{-3}\eta(\sigma)\|_\infty \lesssim \beta^{5/2}(\tau(\sigma))M_1(T).$$

To prove (129) we recall the definition of D_2 and rewrite it as

$$D_2 = \left\{ \left[\frac{\lambda^2}{\lambda_1^2} - 1 \right] \frac{p(a + \frac{1}{2})}{p-1+yby} + \frac{p(a-\alpha)}{p-1+yby} + \frac{p(\frac{\lambda_1^2}{\lambda^2} - 1)yby}{(p-1+zbz)(p-1+yby)} \right. \\ \left. + \frac{p(\alpha - \frac{1}{2})}{p-1+yby} - \frac{2p(\alpha - \frac{1}{2})}{p-1+z\tilde{\beta}z} + \frac{pz(\tilde{\beta} - b)z}{(p-1+zbz)(p-1+z\tilde{\beta}z)} \right\} \eta.$$

Then Equations (93), (96) and the definition of B in (49) imply

$$\|\langle z \rangle^{-3} D_2(\sigma)\|_\infty \leq \beta^{\frac{\kappa}{2}}(\tau(\sigma)) \|\langle z \rangle^{-3} \eta(\sigma)\|_\infty.$$

Using (134) we obtain (129) (recall $\kappa := \min\{\frac{1}{2}, \frac{p-1}{2}\}$).

Now we prove (130). By (132) and the relation between D_3 , F and \mathcal{F} we have

$$\|\langle z \rangle^{-3} D_3(\sigma)\|_\infty \lesssim \|\langle y \rangle^{-3} \mathcal{F}(a, b, c)(\tau(\sigma))\|_\infty$$

which together with (88) implies (130).

Lastly we prove (131). By the relation between D_4 , N and \mathcal{N} and the estimate in (74) we have

$$\begin{aligned} \|\langle z \rangle^{-3} D_4(\sigma)\|_\infty &\lesssim \|\langle y \rangle^{-3} \mathcal{N}(\xi(\tau(\sigma)), b(\tau(\sigma)), c(\tau(\sigma)))\|_\infty \\ &\lesssim \|\langle y \rangle^{-3} \xi(\tau(\sigma))\|_\infty [\xi(\tau(\sigma))\|_\infty + \xi(\tau(\sigma))\|_\infty^{p-1}]. \end{aligned}$$

Using (136) and the definition of M_1 we complete the proof. \square

Below we will need the following lemma. Recall that $S := \sigma(t(T))$.

Lemma 20. *If $A(\tau) \leq \beta^{-\frac{\kappa}{2}}(\tau)$, then for any $c_1, c_2 > 0$ there exists a constant $c(c_1, c_2)$ such that*

$$\int_0^S e^{-c_1(S-\sigma)} \beta^{c_2}(\tau(t(\sigma))) d\sigma \leq c(c_1, c_2) \beta^{c_2}(T). \quad (138)$$

Proof. We use the shorthand $\tau(\sigma) \equiv \tau(t(\sigma))$, where, recall, $t(\sigma)$ is the inverse of $\sigma(t) = \int_0^t \lambda_1^2(k) dk$ and $\tau(t) = \int_0^t \lambda^2(k) dk$. By Proposition 12 we have that $\frac{1}{2} \leq \frac{\lambda}{\lambda_1} \leq 2$ provided that $A(\tau) \leq \beta^{-\frac{\kappa}{2}}(\tau)$. Hence

$$\frac{1}{4} \sigma \leq \tau(\sigma) \leq 4\sigma \quad (139)$$

which implies $\frac{1}{\frac{1}{\beta(0)} + \frac{4p}{(p-1)^2} \tau(\sigma)} \lesssim \frac{1}{\frac{1}{\beta(0)} + \sigma}$. By a direct computation we have

$$\int_0^S e^{-c_1(S-\sigma)} \beta^{c_2}(\tau(\sigma)) d\sigma \leq c(c_1, c_2) \frac{1}{(\frac{1}{\beta(0)} + \frac{4p}{p-1} S)^{c_2}}. \quad (140)$$

Using (139) again we obtain $4S \geq \tau(S) = T \geq \frac{1}{4}S$ which together with (140) implies (138). \square

Recall that $V_\alpha(t, s)$ is the propagator generated by the operator $-P^\alpha L_\alpha P^\alpha$. To estimate the function $P^\alpha \eta$ we rewrite Equation (127) as

$$P^\alpha \eta(S) = V_\alpha(S, 0)P^\alpha \eta(0) + \sum_{n=1}^4 \int_0^S V_\alpha(S, \sigma) P^\alpha D_n(\sigma) d\sigma$$

which implies

$$\|\langle z \rangle^{-3} P^\alpha \eta(S)\|_\infty \leq K_1 + K_2 \quad (141)$$

with

$$K_1 := \|\langle z \rangle^{-3} V_\alpha(S, 0) P^\alpha \eta(0)\|_\infty;$$

$$K_2 := \|\langle z \rangle^{-3} \sum_{n=1}^4 \int_0^S V_\alpha(S, \sigma) P^\alpha D_n(\sigma) d\sigma\|_\infty.$$

Using Theorem 13, equation (134) and the slow decay of $\beta(\tau)$ we obtain

$$K_1 \lesssim e^{-c_0 S} \|\langle z \rangle^{-3} P^\alpha \eta(0)\|_\infty \lesssim e^{-c_0 S} \|\langle z \rangle^{-3} \eta(0)\|_\infty \lesssim \beta^2(T) M_1(0). \quad (142)$$

By Theorem 13, equations (128)-(131) and $\int_0^S e^{-c_0(S-\sigma)} \beta^2(\tau(\sigma)) d\sigma \lesssim \beta^2(T)$ (see Lemma 20) we have

$$K_2 \lesssim \beta^2(T) \{ \beta^{\frac{k}{2}}(0) [1 + M_1(T)A(T) + M_1^2(T) + M_1^p(T)] \\ + [M_2(T)M_1(T) + M_1(T)M_2^{p-1}(T)] \}. \quad (143)$$

Equation (90) and the definitions of S and T imply that $\lambda_1(t(S)) = \lambda(t(T))$, $z = y$, $\eta(S) = \xi(T)$, and $P^\alpha \xi = \xi$, consequently

$$\|\langle z \rangle^{-3} P^\alpha \eta(S)\|_\infty = \|\langle y \rangle^{-3} \xi(T)\|_\infty. \quad (144)$$

Collecting the estimates (141)-(144) and using the definition of M_1 in (49) we have

$$M_1(T) := \sup_{\tau \leq T} \beta^{-2}(\tau) \|\langle y \rangle^{-3} \xi(\tau)\|_\infty$$

$$\lesssim M_1(0) + \beta^{\frac{k}{2}}(0) [1 + M_1(T)A(T) + M_1^2(T) + M_1^p(T)] \\ + M_2(T)M_1(T) + M_1(T)M_2^{p-1}(T)$$

which together with the fact that T is arbitrary implies Equation (53).

□

11 Estimate of M_2 (Equation (54))

The following lemma is proven similarly to the corresponding parts of Lemma 19 and therefore it is presented without a proof.

Lemma 21. *If $A(\tau), B(\tau) \leq \beta^{-\frac{\kappa}{2}}(\tau)$ and $b_0 \ll 1$ and $D_k(\sigma)$, $k = 2, 3, 4$, are the same as in Lemma 19, then*

$$\|D_2(\sigma)\|_\infty \lesssim \beta^{\frac{\kappa}{2}}(\tau(\sigma))[\beta^{1/2}(\tau(\sigma))M_1(T) + M_2(T)]; \quad (145)$$

$$\|D_3(\sigma)\|_\infty \lesssim \beta^{\min\{1, 2p-1\}}(\tau(\sigma))[1 + M_1(T)(1 + A(T)) + M_1^2(T) + M_1^p(T)]; \quad (146)$$

$$\|D_4(\sigma)\|_\infty \lesssim \beta(\tau(\sigma))M_1^2(T) + M_2^2(T) + \beta^{p/2}(\tau(\sigma))M_1^p(T) + M_2^p(T). \quad (147)$$

To estimate M_2 it is convenient to treat the z -dependent part of the potential in (92) as a perturbation. Let the operator L_0 be the same as in (91). Rewrite (91) to have

$$\eta(S) = e^{-(L_0 + \frac{2p\alpha}{p-1})S}\eta(0) + \int_0^S e^{-(L_0 + \frac{2p\alpha}{p-1})(S-\sigma)}(V_2\eta(\sigma) + \sum_{k=2}^4 D_k(\sigma))d\sigma, \quad (148)$$

where, recall $S := \sigma(t(T))$, V_2 is the operator given by

$$V_2 := \frac{2p\alpha}{p-1 + z\tilde{\beta}(\tau(\sigma))z},$$

and the terms D_n , $n = 2, 3, 4$, are the same as in (127). Lemma 14 implies that

$$\|e^{-(L_0 + \frac{2p\alpha}{p-1})s}g\|_\infty = e^{-\frac{2p\alpha}{p-1}s}\|e^{-L_0s}g\|_\infty \lesssim e^{-\frac{2\alpha}{p-1}s}\|g\|_\infty$$

for any function g and time $s \geq 0$. Hence we have

$$\|\eta(S)\|_\infty \lesssim K_0 + K_1 + K_2 \quad (149)$$

where the functions K_i are given by

$$\begin{aligned} K_0 &:= e^{-\frac{2\alpha}{p-1}S}\|\eta(0)\|_\infty; \\ K_1 &:= \int_0^S e^{-\frac{2\alpha}{p-1}(S-\sigma)}\|V_2\eta(\sigma)\|_\infty d\sigma, \\ K_2 &:= \sum_{n=2}^4 \int_0^S e^{-\frac{2\alpha}{p-1}(S-\sigma)}\|D_n\|_\infty d\sigma. \end{aligned}$$

We estimate the K_n 's, $n = 0, 1, 2$.

(K0) We start with K_0 . By (133) and the decay of $e^{-\frac{2\alpha}{p-1}S}$ we have

$$K_0 \lesssim M_2(0) + \beta^{1/2}(0)M_1(0). \quad (150)$$

(K1) By the definition of V_2 we have

$$\|V_2\eta(\sigma)\|_\infty \lesssim \left\| \frac{1}{\beta(\tau(\sigma))} \langle z \rangle^{-2} \eta(\sigma) \right\|_\infty.$$

Moreover by the relation between ξ and η in Equation (90) and Proposition 12 we have

$$\begin{aligned} \max_{0 \leq \sigma \leq S} \|V_2 \eta(\sigma)\|_\infty &\lesssim \max_{0 \leq \tau \leq T} \left\| \frac{1}{\beta} \langle y \rangle^{-2} \xi(\tau) \right\|_\infty \\ &\leq \max_{0 \leq \tau \leq T} \frac{1}{\beta} (\|\langle y \rangle^{-3} \xi(\tau)\|_\infty)^{\frac{2}{3}} (\xi(\tau)\|_\infty)^{\frac{1}{3}} \\ &\leq \beta^{\frac{1}{3}}(0) M_1^{\frac{2}{3}}(T) M_2^{\frac{1}{3}}(T). \end{aligned}$$

Therefore we obtain

$$\begin{aligned} K_1 &\lesssim \max_{0 \leq \sigma \leq S} \|V_2 \eta(\sigma)\|_\infty \int_0^S e^{-\frac{2\alpha}{p-1}(S-\sigma)} d\sigma \\ &\lesssim \beta^{\frac{1}{3}}(0) M_1^{\frac{2}{3}}(T) M_2^{\frac{1}{3}}(T). \end{aligned} \tag{151}$$

(K2) By the definitions of D_k , $k = 2, 3, 4$, and Equations (145)-(147) we have

$$\begin{aligned} \sum_{k=2}^4 \|D_k(\sigma)\|_\infty &\lesssim \beta^{\frac{2}{3}}(\tau(\sigma)) [1 + M_2(T) + M_1(T)A(T) + M_1^2(T) + M_1^p(T)] \\ &\quad + M_2^2(T) + M_2^p(T) \end{aligned}$$

and consequently

$$\begin{aligned} K_2 &\lesssim \beta^{\frac{2}{3}}(0) [1 + M_2(T) + M_1(T)A(T) + M_1^2(T) + M_1^p(T)] \\ &\quad + M_2^2(T) + M_2^p(T). \end{aligned} \tag{152}$$

Collecting the estimates (149)-(152) we have

$$\begin{aligned} \|\eta(S)\|_\infty &\lesssim M_2(0) + \beta^{1/2}(0) M_1(0) + \beta^{\frac{1}{3}}(0) M_1^{\frac{2}{3}}(T) M_2^{\frac{1}{3}}(T) \\ &\quad + \beta^{\frac{2}{3}}(0) [1 + M_2(T) + M_1(T)A(T) + M_1^2(T) + M_1^p(T)] + M_2^2(T) + M_2^p(T). \end{aligned} \tag{153}$$

The relation between ξ and η in Equation (90) implies

$$\|\xi(T)\|_\infty = \|\eta(S)\|_\infty$$

which together with (153) gives

$$\begin{aligned} M_2(T) &\lesssim M_2(0) + \beta^{1/2}(0) M_1(0) + \beta^{\frac{1}{3}}(0) M_1^{\frac{2}{3}}(T) M_2^{\frac{1}{3}}(T) + M_2^2(T) + M_2^p(T) \\ &\quad + \beta^{\frac{2}{3}}(0) [1 + M_2(T) + M_1(T)A(T) + M_1^2(T) + M_1^p(T)]. \end{aligned}$$

Since T is an arbitrary time, the proof of the estimate (54) for M_2 is complete.

Appendix 1: Feynmann-Kac Formula

In this appendix we present, for the reader's convenience, a proof of the Feynmann-Kac formula $U(x, y) = U_0(x, y)E(x, y)$ and the estimate (111) used in section 9 (cf. [2, 6]). For stochastic calculus proofs of similar formulae see [7, 22, 25, 26, 42].

Let $L_0 := -\Delta_y + \frac{\alpha^2}{4}|y|^2 - \frac{\alpha}{2}$ and $L := L_0 + V$ where V is a multiplication operator by a function $V(y, \tau)$, which is bounded and Lipschitz continuous in τ . Let $U(\tau, \sigma)$ and $U_0(\tau, \sigma)$ be the propagators generated by the operators $-L$ and $-L_0$, respectively. The integral kernels of these operators will be denoted by $U(\tau, \sigma)(x, y)$ and $U_0(\tau, \sigma)(x, y)$.

Theorem 22. *The integral kernel of $U(\tau, \sigma)$ can be represented as*

$$U(\tau, \sigma)(x, y) = U_0(\tau, \sigma)(x, y) \int e^{\int_{\sigma}^{\tau} V(\omega_0(s) + \omega(s), s) ds} d\mu(\omega) \quad (154)$$

where $d\mu(\omega)$ is a probability measure (more precisely, a conditional harmonic oscillator, or Ornstein-Uhlenbeck, probability measure) on the continuous paths $\omega : [\sigma, \tau] \rightarrow \mathbb{R}^n$ with $\omega(\sigma) = \omega(\tau) = 0$, and $\omega_0(\cdot)$ is the path defined as

$$\omega_0(s) = e^{\alpha(\tau-s)} \frac{e^{2\alpha\sigma} - e^{2\alpha s}}{e^{2\alpha\sigma} - e^{2\alpha\tau}} x + e^{\alpha(\sigma-s)} \frac{e^{2\alpha\tau} - e^{2\alpha s}}{e^{2\alpha\tau} - e^{2\alpha\sigma}} y. \quad (155)$$

Remark 1. $d\mu(\omega)$ is the Gaussian measure with mean zero and covariance $(-\partial_s^2 + \alpha^2)^{-1}$, normalized to 1. The path $\omega_0(s)$ solves the boundary value problem

$$(-\partial_s^2 + \alpha^2)\omega_0 = 0 \text{ with } \omega(\sigma) = y \text{ and } \omega(\tau) = x. \quad (156)$$

Below we will also deal with the normalized Gaussian measure $d\mu_{xy}(\omega)$ with mean $\omega_0(s)$ and covariance $(-\partial_s^2 + \alpha^2)^{-1}$. This is a conditional Ornstein-Uhlenbeck probability measure on continuous paths $\omega : [\sigma, \tau] \rightarrow \mathbb{R}^n$ with $\omega(\sigma) = y$ and $\omega(\tau) = x$ (see e.g. [22, 25, 42]).

Now, assume in addition that the function $V(y, \tau)$ satisfies the estimates

$$V \leq 0 \text{ and } |\partial_y V(y, \tau)| \lesssim \beta^{\frac{1}{2}}(\tau) \quad (157)$$

where $\beta(\tau)$ is a positive function. Then Theorem 22 implies Equation (111) by the following corollary.

Corollary 23. *Under (157),*

$$|\partial_y \int e^{\int_{\sigma}^{\tau} V(\omega_0(s) + \omega(s), s) ds} d\mu(\omega)| \lesssim |\tau - \sigma| \sup_{\sigma \leq s \leq \tau} \beta^{\frac{1}{2}}(\tau)$$

Proof. By Fubini's theorem

$$\partial_y \int e^{\int_{\sigma}^{\tau} V(\omega_0(s) + \omega(s), s) ds} d\mu(\omega) = \int \partial_y [\int_{\sigma}^{\tau} V(\omega_0(s) + \omega(s), s) ds] e^{\int_{\sigma}^{\tau} V(\omega_0(s) + \omega(s), s) ds} d\mu(\omega)$$

Equation (157) implies

$$|\partial_y \int_{\sigma}^{\tau} V(\omega_0(s) + \omega(s), s) ds| \leq |\tau - \sigma| \sup_{\sigma \leq s \leq \tau} \beta^{\frac{1}{2}}(\tau), \text{ and } e^{\int_{\sigma}^{\tau} V(\omega_0(s) + \omega(s), s) ds} \leq 1.$$

Thus

$$|\partial_y \int e^{\int_{\sigma}^{\tau} V(\omega_0(s) + \omega(s), s) ds} d\mu(\omega)| \lesssim |\tau - \sigma| \sup_{\sigma \leq s \leq \tau} \beta^{\frac{1}{2}}(\tau) \int d\mu(\omega) = |\tau - \sigma| \sup_{\sigma \leq s \leq \tau} \beta^{\frac{1}{2}}(\tau)$$

to complete the proof. \square

Proof of Theorem 22. We begin with the following extension of the Ornstein-Uhlenbeck process-based Feynman-Kac formula to time-dependent potentials:

$$U(\tau, \sigma)(x, y) = U_0(\tau, \sigma)(x, y) \int e^{\int_{\sigma}^{\tau} V(\omega(s), s) ds} d\mu_{xy}(\omega). \quad (158)$$

where $d\mu_{xy}(w)$ is the conditional Ornstein-Uhlenbeck probability measure described in Remark 1 above. This formula can be proven in the same way as the one for time independent potentials (see [22], Equation (3.2.8)), i.e. by using the Kato-Trotter formula and evaluation of Gaussian measures on cylindrical sets. Since its proof contains a slight technical wrinkle, for the reader's convenience we present it below.

Now changing the variable of integration in (158) as $\omega = \omega_0 + \tilde{\omega}$, where $\tilde{\omega}(s)$ is a continuous path with boundary conditions $\tilde{\omega}(\sigma) = \tilde{\omega}(\tau) = 0$, using the translational change of variables formula $\int f(\omega) d\mu_{xy}(\omega) = \int f(\omega_0 + \tilde{\omega}) d\mu(\tilde{\omega})$, which can be proven by taking $f(\omega) = e^{i\langle \omega, \zeta \rangle}$ and using (156) (see [22], Equation (9.1.27)) and omitting the tilde over ω we arrive at (154). \square

There are at least three standard ways to prove (158): by using the Kato-Trotter formula, by expanding both sides of the equation in V and comparing the resulting series term by term and by using Ito's calculus (see [26, 42, 41, 22]). The first two proofs are elementary but involve tedious estimates while the third proof is based on a fair amount of stochastic calculus. For the reader's convenience, we present the first elementary proof of (158).

Before starting proving (158) we establish an auxiliary result. We define the operator \mathcal{K} as

$$\mathcal{K}(\sigma, \delta) := \int_0^{\delta} U_0(\sigma + \delta, \sigma + s) V(\sigma + s, \cdot) U_0(\sigma + s, \sigma) ds - U_0(\sigma + \delta, \sigma) \int_0^{\delta} V(\sigma + s, \cdot) ds \quad (159)$$

Lemma 24. *For any $\xi \in \mathcal{C}_0^{\infty}$ we have, as $\delta \rightarrow 0^+$,*

$$\sup_{0 \leq \sigma \leq \tau} \left\| \frac{1}{\delta} \mathcal{K}(\sigma, \delta) U(\sigma, 0) \xi \right\|_2 \rightarrow 0. \quad (160)$$

Proof. If the potential term, V , is independent of τ , then the proof is standard (see, e.g. [41]). We use the property that the function V is Lipschitz continuous in time τ to prove (160). The operator \mathcal{K} can be further decomposed as

$$\mathcal{K}(\sigma, \delta) = \mathcal{K}_1(\sigma, \delta) + \mathcal{K}_2(\sigma, \delta)$$

with

$$\mathcal{K}_1(\sigma, \delta) := \int_0^{\delta} U_0(\sigma + \delta, \sigma + s) V(\sigma, \cdot) U_0(\sigma + s, \sigma) ds - \delta U_0(\sigma + \delta, \sigma) V(\sigma, \cdot)$$

and

$$\mathcal{K}_2(\sigma, \delta) := \int_0^{\delta} U_0(\sigma + \delta, \sigma + s) [V(\sigma + s, \cdot) - V(\sigma, \cdot)] U_0(\sigma + s, \sigma) ds - U_0(\sigma + \delta, \sigma) \int_0^{\delta} [V(\sigma + s, \cdot) - V(\sigma, \cdot)] ds.$$

Since $U_0(\tau, \sigma)$ are uniformly L^2 -bounded and V is bounded, we have $U(\tau, \sigma)$ is uniformly L^2 -bounded. This together with the fact that the function $V(\tau, y)$ is Lipschitz continuous in τ implies that

$$\|\mathcal{K}_2(\sigma, \delta)\|_{L^2 \rightarrow L^2} \lesssim 2 \int_0^{\delta} s ds = \delta^2.$$

We rewrite $\mathcal{K}_1(\sigma, \delta)$ as

$$\mathcal{K}_1(\sigma, \delta) = \int_0^\delta U_0(\sigma + \delta, \sigma + s) \{V(\sigma, \cdot) [U_0(\sigma + s, \sigma) - 1] - [U_0(\sigma + s, \sigma) - 1] V(\sigma, \cdot)\} ds.$$

Let $\xi(\sigma) = U(\sigma, 0)\xi$. We claim that for a fixed $\sigma \in [0, \tau]$,

$$\|\mathcal{K}_1(\sigma, \delta)\xi(\sigma)\|_2 = o(\delta). \quad (161)$$

Indeed, the fact $\xi_0 \in \mathcal{C}_0^\infty$ implies that $L_0\xi(\sigma)$, $L_0V(\sigma)\xi(\sigma) \in L^2$. Consequently (see [40])

$$\lim_{s \rightarrow 0^+} \frac{(U_0(\sigma + s, \sigma) - 1)g}{s} \rightarrow L_0g,$$

for $g = \xi(\sigma)$ or $V(\sigma, y)\xi(\sigma)$ which implies our claim. Since the set of functions $\{\xi(\sigma) | \sigma \in [0, \tau]\} \subset L_0L^2$ is compact and $\|\frac{1}{\delta}\mathcal{K}_1(\sigma, \delta)\|_{L^2 \rightarrow L^2}$ is uniformly bounded, we have (161) as $\delta \rightarrow 0$ uniformly in $\sigma \in [0, \tau]$.

Collecting the estimates on the operators \mathcal{K}_i , $i = 1, 2$, we arrive at (160). \square

Lemma 25. *Equation (158) holds.*

Proof. In order to simplify our notation, in the proof that follows we assume, without losing generality, that $\sigma = 0$. We divide the proof into two parts. First we prove that for any fixed $\xi \in \mathcal{C}_0^\infty$ the following Kato-Trotter type formula holds

$$U(\tau, 0)\xi = \lim_{m \rightarrow \infty} \prod_{0 \leq k \leq m-1} U_0\left(\frac{k+1}{m}\tau, \frac{k}{m}\tau\right) e^{\int_{\frac{k\tau}{m}}^{\frac{(k+1)\tau}{m}} V(y,s)ds} \xi \quad (162)$$

in the L^2 space. We start with the formula

$$\begin{aligned} & U(\tau, 0) - \prod_{0 \leq k \leq m-1} U_0\left(\frac{k+1}{m}\tau, \frac{k}{m}\tau\right) e^{\int_{\frac{k\tau}{m}}^{\frac{(k+1)\tau}{m}} V(y,s)ds} \\ &= \prod_{0 \leq k \leq m-1} U\left(\frac{k+1}{m}\tau, \frac{k}{m}\tau\right) - \prod_{0 \leq k \leq m-1} U_0\left(\frac{k+1}{m}\tau, \frac{k}{m}\tau\right) e^{\int_{\frac{k\tau}{m}}^{\frac{(k+1)\tau}{m}} V(y,s)ds} \\ &= \sum_{0 \leq j \leq m} \prod_{j \leq k \leq m-1} U_0\left(\frac{k+1}{m}\tau, \frac{k}{m}\tau\right) e^{\int_{\frac{k\tau}{m}}^{\frac{(k+1)\tau}{m}} V(y,s)ds} A_j U\left(\frac{j}{m}\tau, 0\right) \end{aligned}$$

with the operator

$$A_j := U_0\left(\frac{j+1}{m}\tau, \frac{j}{m}\tau\right) e^{\int_{\frac{j\tau}{m}}^{\frac{(j+1)\tau}{m}} V(y,s)ds} - U\left(\frac{j+1}{m}\tau, \frac{j}{m}\tau\right).$$

We observe that $\|U_0(\tau, \sigma)\|_{L^2 \rightarrow L^2} \leq 1$, and moreover, by the boundness of V , the operator $U(\tau, \sigma)$ is

uniformly bounded in τ and σ in any compact set. Consequently

$$\begin{aligned}
& \| [U(\tau, 0) - \prod_{0 \leq k \leq m-1} U_0(\frac{k+1}{m}\tau, \frac{k}{m}\tau) e^{\int_{\frac{k\tau}{m}}^{\frac{(k+1)\tau}{m}} V(y,s)ds}] \xi \|_2 \\
& \leq \max_j m \| \prod_{j \leq k \leq m-1} U_0(\frac{k+1}{m}\tau, \frac{k}{m}\tau) e^{\int_{\frac{k\tau}{m}}^{\frac{(k+1)\tau}{m}} V(y,s)ds} A_j U(\frac{j}{m}\tau, 0) \xi \|_2 \\
& \lesssim m \max_j \| A_j + \mathcal{K}(\frac{k}{m}\tau, \frac{1}{m}\tau) \|_{L^2 \rightarrow L^2} + \max_j m \| \mathcal{K}(\frac{j}{m}\tau, \frac{1}{m}\tau) U(\frac{j}{m}\tau, 0) \xi \|_2
\end{aligned} \tag{163}$$

where, recall the definition of \mathcal{K} from (159). Now we claim that

$$\| A_j + \mathcal{K}(\frac{k}{m}\tau, \frac{1}{m}\tau) \|_{L^2 \rightarrow L^2} \lesssim \frac{1}{m^2}. \tag{164}$$

Indeed, by the Duhamel principle we have

$$U(\frac{j+1}{m}\tau, \frac{j}{m}\tau) = U_0(\frac{j+1}{m}\tau, \frac{j}{m}\tau) + \int_{\frac{j\tau}{m}}^{\frac{j+1}{m}\tau} U_0(\frac{j+1}{m}\tau, s) V(y, s) U(s, \frac{j}{m}\tau) ds.$$

Iterating this equation on $U(s, \frac{k}{m}\tau)$ and using the fact that $U(s, t)$ is uniformly bounded if s, t is on a compact set, we obtain

$$\| U(\frac{j+1}{m}\tau, \frac{j}{m}\tau) - U_0(\frac{j+1}{m}\tau, \frac{j}{m}\tau) - \int_0^{\frac{1}{m}\tau} U_0(\frac{j+1}{m}\tau, s) V(y, s) U_0(s, \frac{j}{m}\tau) ds \|_{L^2 \rightarrow L^2} \lesssim \frac{1}{m^2}.$$

On the other hand we expand $e^{\int_{\frac{j\tau}{m}}^{\frac{(j+1)\tau}{m}} V(y,s)ds}$ and use the fact that V is bounded to get

$$\| U_0(\frac{j+1}{m}\tau, \frac{j}{m}\tau) e^{\int_{\frac{j\tau}{m}}^{\frac{(j+1)\tau}{m}} V(y,s)ds} - U_0(\frac{j+1}{m}\tau, \frac{j}{m}\tau) - U_0(\frac{j+1}{m}\tau, \frac{j}{m}\tau) \int_{\frac{j\tau}{m}}^{\frac{(j+1)\tau}{m}} V(y,s)ds \|_{L^2 \rightarrow L^2} \lesssim \frac{1}{m^2}.$$

By the definition of \mathcal{K} and A_j we complete the proof of (164). Equations (160), (163) and (164) imply (162). This completes the first step.

In the second step we compute the integral kernel, $G_m(x, y)$, of the operator

$$G_m := \prod_{0 \leq k \leq m-1} U_0(\frac{k+1}{m}\tau, \frac{k}{m}\tau) e^{\int_{\frac{k\tau}{m}}^{\frac{(k+1)\tau}{m}} V(\cdot, s)ds}$$

in (162). By the definition, $G_m(x, y)$ can be written as

$$G_m(x, y) = \int \cdots \int \prod_{0 \leq k \leq m-1} U_{\frac{\tau}{m}}(x_{k+1}, x_k) e^{\int_{\frac{k\tau}{m}}^{\frac{(k+1)\tau}{m}} V(x_k, s)ds} dx_1 \cdots dx_{m-1} \tag{165}$$

with $x_m := x$, $x_0 := y$ and $U_\tau(x, y) \equiv U_0(0, \tau)(x, y)$ is the integral kernel of the operator $U_0(\tau, 0) = e^{-L_0\tau}$. We rewrite (165) as

$$G_m(x, y) = U_\tau(x, y) \int e^{\sum_{k=0}^{m-1} \int_{\frac{k\tau}{m}}^{\frac{(k+1)\tau}{m}} V(x_k, s) ds} d\mu_m(x_1, \dots, x_m), \quad (166)$$

where

$$d\mu_m(x_1, \dots, x_m) := \frac{\prod_{0 \leq k \leq m-1} U_{\frac{\tau}{m}}(x_{k+1}, x_k)}{U_\tau(x, y)} dx_1 \dots dx_{m-1}.$$

Since $G_m(x, y)|_{V=0} = U_\tau(x, y)$ we have that $\int d\mu_m(x_1, \dots, x_m) = 1$. Let $\Delta := \Delta_1 \times \dots \times \Delta_m$, where Δ_j is an interval in \mathbb{R} . Define a cylindrical set

$$P_\Delta^m := \{\omega : [0, \tau] \rightarrow \mathbb{R}^n \mid \omega(0) = y, \omega(\tau) = x, \omega(k\tau/m) \in \Delta_k, 1 \leq k \leq m-1\}.$$

By the definition of the measure $d\mu_{xy}(\omega)$, we have $\mu_{xy}(P_\Delta^m) = \int_\Delta d\mu_m(x_1, \dots, x_m)$. Thus, we can rewrite (166) as

$$G_m(x, y) = U_\tau(x, y) \int e^{\sum_{k=0}^{m-1} \int_{\frac{k\tau}{m}}^{\frac{(k+1)\tau}{m}} V(\omega(\frac{k\tau}{m}), s) ds} d\mu_{xy}(\omega), \quad (167)$$

By the dominated convergence theorem the integral on the right hand side of (167) converges in the sense of distributions as $m \rightarrow \infty$ to the integral on the right hand side of (158). Since the left hand side of (167) converges to the left hand side of (158), also in the sense of distributions (which follows from the fact that G_m converges in the operator norm on L^2 to $U(\tau, \sigma)$), (158) follows. \square

Note that on the level of finite dimensional approximations the change of variables formula can be derived as follows. It is tedious, but not hard, to prove that

$$\prod_{0 \leq k \leq m-1} U_m(x_{k+1}, x_k) = e^{-\alpha \frac{(x - e^{-\alpha\tau}y)^2}{2(1 - e^{-2\alpha\tau})}} \prod_{0 \leq k \leq m-1} U_m(y_{k+1}, y_k)$$

with $y_k := x_k - \omega_0(\frac{k}{m}\tau)$. By the definition of $\omega_0(s)$ and the relations $x_0 = y$ and $x_m = x$ we have

$$G_m(x, y) = U_\tau(x, y) G_m^{(1)}(x, y) \quad (168)$$

where

$$G_m^{(1)}(x, y) := \frac{1}{4\pi\sqrt{\alpha}(1 - e^{-2\alpha\tau})} \int \dots \int \prod_{0 \leq k \leq m-1} U_m(y_{k+1}, y_k) e^{\int_{\frac{k\tau}{m}}^{\frac{(k+1)\tau}{m}} V(y_k + \omega_0(\frac{k\tau}{m}), s) ds} dy_1 \dots dy_{m-1}. \quad (169)$$

Since $\lim_{m \rightarrow \infty} G_m \xi$ exists by (160), we have $\lim_{m \rightarrow \infty} G_m^{(1)} \xi$ (in the weak limit) exists also. As shown in [22],

$\lim_{m \rightarrow \infty} G_m^{(1)} = \int e^{\int_0^\tau V(\omega_0(s) + \omega(s), s) ds} d\mu(\omega)$ with $d\mu$ being the (conditional) Ornstein-Uhlenbeck measure on the set of path from 0 to 0. This completes the derivation of the change of variables formula.

Remark 2. In fact, Equations (162), (168) and (169) suffice to prove the estimate in Corollary 23.

Appendix 2: Computations and Proofs

Equation (29): Computation of A_1 . Here through some examples we show how to compute the matrix A_1 . We have

$$\begin{aligned}
\langle \partial_a V_\mu, \varphi_{az}^{ij} \rangle &= \lambda^{-n+\frac{2}{p-1}} \langle \partial_a V_{ab}, \phi_a^{(ij)} \rangle \\
&= \frac{\lambda^{-n+\frac{2}{p-1}}}{p-1} \left(\frac{a+\frac{1}{2}}{p-1} \right)^{\frac{1}{p-1}} \frac{1}{a+\frac{1}{2}} \int e^{-\frac{a}{4}|y|^2} \phi_a^{(ij)} dy + O(\|b\|) \\
&= \frac{\lambda^{-n+\frac{2}{p-1}}}{p-1} \left(\frac{a+\frac{1}{2}}{p-1} \right)^{\frac{1}{p-1}} \frac{1}{a+\frac{1}{2}} \left(\frac{2\pi}{a} \right)^{n/2} \delta_{ij} + O(\|b\|), \\
\langle \partial_{b_{ii}} V_\mu, \varphi_{az}^{jj} \rangle &= \lambda^{-n+\frac{2}{p-1}} \langle \partial_{b_{ii}} V_{ab}, e^{-\frac{a}{4}|y|^2} \phi_a^{(jj)} \rangle \\
&= -\frac{\lambda^{-n+\frac{2}{p-1}}}{(p-1)^2} \left(\frac{a+\frac{1}{2}}{p-1} \right)^{\frac{1}{p-1}} \int y_i^2 e^{-\frac{a}{4}|y|^2} \phi_a^{(jj)} dy + O(\|b\|) \\
&= \begin{cases} -\frac{\lambda^{-n+\frac{2}{p-1}}}{(p-1)^2} \left(\frac{a+\frac{1}{2}}{p-1} \right)^{\frac{1}{p-1}} \frac{3}{a} \left(\frac{2\pi}{a} \right)^{n/2} + O(\|b\|), & \text{if } i = j, \\ -\frac{\lambda^{-n+\frac{2}{p-1}}}{(p-1)^2} \left(\frac{a+\frac{1}{2}}{p-1} \right)^{\frac{1}{p-1}} \frac{1}{a} \left(\frac{2\pi}{a} \right)^{n/2} + O(\|b\|), & \text{if } i \neq j. \end{cases}
\end{aligned}$$

Similarly we can compute all the other entries.

Derivation of Equation (65)-(71). Let $v = V_{ab} + \xi$, then we have

$$\begin{aligned}
\partial_\tau v &= \frac{1}{p-1} \left(\frac{c}{p-1+yby} \right)^{\frac{1}{p-1}-1} \frac{c_\tau (p-1+yby) - cyb_\tau y}{(p-1+yby)^2} + \xi_\tau, \\
\partial_{y_i} v &= -\frac{1}{p-1} \left(\frac{c}{p-1+yby} \right)^{\frac{1}{p-1}} \frac{2 \sum_j b_{ij} y_j}{p-1+yby} + \partial_{y_i} \xi, \\
\partial_{y_i}^2 v &= \frac{1}{(p-1)^2} \left(\frac{c}{p-1+yby} \right)^{\frac{1}{p-1}} \left(\frac{2 \sum_j b_{ij} y_j}{p-1+yby} \right)^2 - \frac{1}{p-1} \left(\frac{c}{p-1+yby} \right)^{\frac{1}{p-1}} \frac{2b_{ii}(p-1+yby) - (2 \sum_j b_{ij} y_j)^2}{(p-1+yby)^2} \\
&\quad + \partial_{y_i}^2 \xi.
\end{aligned} \tag{170}$$

Plugging (170) into (20) we obtain

$$\begin{aligned}
&\partial_\tau \xi + \frac{c_\tau (p-1+yby) - cyb_\tau y}{(p-1)(p-1+yby)c} V_{ab} \\
&= \frac{4 \sum_i (\sum_j b_{ij} y_j)^2}{(p-1)^2 (p-1+yby)^2} V_{ab} - \sum_i \frac{2b_{ii}(p-1+yby) - 4(\sum_j b_{ij} y_j)^2}{(p-1)(p-1+yby)^2} V_{ab} \\
&\quad + \Delta \xi + \frac{2ayby}{(p-1)(p-1+yby)} V_{ab} - a \sum_i y_i \partial_{y_i} \xi - \frac{2a}{p-1} V_{ab} - \frac{2a}{p-1} \xi + |V_{ab} + \xi|^{p-1} (V_{ab} + \xi).
\end{aligned}$$

It follows that

$$\begin{aligned}
\partial_\tau \xi &= (\Delta - ay \cdot \partial_y - \frac{2a}{p-1} + \frac{pc}{p-1+yby}) \xi + |V_{ab} + \xi|^{p-1} (V_{ab} + \xi) - V_{ab}^p - pV_{ab}^{p-1} \xi \\
&\quad + \left(\frac{c}{p-1+yby} - \frac{2a}{p-1} + \frac{2ayby}{(p-1)(p-1+yby)} + \frac{4p \sum_i (\sum_j b_{ij} y_j)^2}{(p-1)^2 (p-1+yby)^2} - \frac{2 \sum_i b_{ii}}{(p-1)(p-1+yby)} - \frac{c_\tau/c}{p-1} + \frac{yb_\tau y}{(p-1)(p-1+yby)} \right) V_{ab}.
\end{aligned}$$

Rearranging the terms on the r.h.s. we obtain the equations (65)-(71) for ξ .

Proof of Lemma 16. We prove this result by induction in the dimension n . For $n = 1$, the result is straightforward since $1 = P_0^{(1)} + P_1^{(1)} + P_2^{(1)} + P_3^{(1)}$.

Assume the statement of the lemma is true for all dimensions $m \leq n-1$ and we will prove it for dimension n . By symmetry we only need to prove it for the case k . We have by assumption

$$1 = \sum_{\vec{i} \in J'_1} P_{\vec{i}} \tag{171}$$

where $J'_1 \subset I_1^{(n-1)}$. We claim the following relations

$$P_0^{(n)} = \sum_{\vec{i} \in J'_1} P_{\vec{i}} P_0^{(n)}, \quad (172)$$

$$\begin{aligned} P_1^{(n)} &= P_0^{(1)} \dots P_0^{(n-1)} P_1^{(n)} + \sum_{j=1}^{n-1} P_0^{(1)} \dots P_0^{(j-1)} P_{1'}^{(j)} P_0^{(j+1)} \dots P_0^{(n-1)} P_1^{(n)} \\ &= P_0^{(1)} \dots P_0^{(n-1)} P_1^{(n)} + \sum_{k \leq l} P_0^{(1)} \dots P_0^{(k-1)} P_{1'}^{(k)} P_0^{(k+1)} \dots P_0^{(l-1)} P_{1'}^{(l)} P_0^{(l+1)} \dots P_0^{(n-1)} P_1^{(n)} \\ &\quad + \sum_{k=1}^{n-1} \sum_{l=1, 2'} P_0^{(1)} \dots P_0^{(k-1)} P_l^{(k)} P_0^{(k+1)} \dots P_0^{(n-1)} P_1^{(n)}, \end{aligned} \quad (173)$$

$$WeP_2^{(n)} = P_0^{(1)} \dots P_0^{(n-1)} P_2^{(n)} + \sum_{j=1}^{n-1} P_0^{(1)} \dots P_0^{(j-1)} P_{1'}^{(j)} P_0^{(j+1)} \dots P_0^{(n-1)} P_2^{(n)}, \quad (174)$$

$$P_3^{(n)} = P_0^{(1)} \dots P_0^{(n-1)} P_3^{(n)}. \quad (175)$$

In fact, (172) follows directly from (171), and (175) is trivial. Moreover, using the second relation in (102) we obtain

$$\begin{aligned} 1 &= P_0^{(1)} \dots P_0^{(n-1)} = P_0^{(1)} \dots P_0^{(n-2)} (P_0^{(n-1)} + P_{1'}^{(n-1)}) \\ &= P_0^{(1)} \dots P_0^{(n-2)} P_0^{(n-1)} + P_0^{(1)} \dots P_0^{(n-2)} P_{1'}^{(n-1)} \\ &= P_0^{(1)} \dots P_0^{(n-3)} (P_0^{(n-2)} + P_{1'}^{(n-2)}) P_0^{(n-1)} + P_0^{(1)} \dots P_0^{(n-2)} P_{1'}^{(n-1)} \\ &= \dots \\ &= P_0^{(1)} \dots P_0^{(n-1)} + \sum_{j=1}^{n-1} P_0^{(1)} \dots P_0^{(j-1)} P_{1'}^{(j)} P_0^{(j+1)} \dots P_0^{(n-1)}, \end{aligned} \quad (176)$$

and

$$\begin{aligned} P_0^{(1)} \dots P_0^{(j-1)} P_{1'}^{(j)} &= P_0^{(1)} \dots P_0^{(j-2)} (P_0^{(j-1)} + P_{1'}^{(j-1)}) P_{1'}^{(j)} \\ &= P_0^{(1)} \dots P_0^{(j-2)} P_0^{(j-1)} P_{1'}^{(j)} + P_0^{(1)} \dots P_0^{(j-2)} P_{1'}^{(j-1)} P_{1'}^{(j)} \\ &= P_0^{(1)} \dots P_0^{(j-3)} (P_0^{(j-2)} + P_{1'}^{(j-2)}) P_0^{(j-1)} P_{1'}^{(j)} + P_0^{(1)} \dots P_0^{(j-2)} P_{1'}^{(j-1)} P_{1'}^{(j)} \\ &= \dots \\ &= P_0^{(1)} \dots P_0^{(j-1)} P_{1'}^{(j)} + \sum_{k < j} P_0^{(1)} \dots P_0^{(k-1)} P_{1'}^{(k)} P_0^{(k+1)} \dots P_0^{(j-1)} P_{1'}^{(j)}. \end{aligned} \quad (177)$$

(174) follows readily from (176). Finally, using (176) and (177) we arrive at (173). Thus by (172)-(175) and the relation $1 = P_0^{(n)} + P_1^{(n)} + P_2^{(n)} + P_3^{(n)}$ we find

$$\begin{aligned} 1 &= \sum_{\vec{i} \in J'_1} P_{\vec{i}} P_0^{(n)} + P_0^{(1)} \dots P_0^{(n-1)} P_1^{(n)} + \sum_{k < l} P_0^{(1)} \dots P_0^{(k-1)} P_{1'}^{(k)} P_0^{(k+1)} \dots P_0^{(l-1)} P_{1'}^{(l)} P_0^{(l+1)} \dots P_0^{(n-1)} P_1^{(n)} \\ &\quad + \sum_{k=1}^{n-1} \sum_{l=1, 2'} P_0^{(1)} \dots P_0^{(k-1)} P_l^{(k)} P_0^{(k+1)} \dots P_0^{(n-1)} P_1^{(n)} + P_0^{(1)} P_0^{(2)} \dots P_0^{(n-1)} P_2^{(n)} \\ &\quad + \sum_{j=1}^{n-1} P_0^{(1)} \dots P_0^{(j-1)} P_{1'}^{(j)} P_0^{(j+1)} \dots P_0^{(n-1)} P_2^{(n)} + P_0^{(1)} \dots P_0^{(n-1)} P_3^{(n)}. \end{aligned}$$

Therefore we obtain $1 = \sum_{\vec{i} \in J_n} P_{\vec{i}}$, where

$$\begin{aligned} J_n &= \{ \vec{i} = (i_1, \dots, i_{n-1}, 0) \mid (i_1, \dots, i_{n-1}) \in J'_1 \} \cup \{ (0, \dots, 0, k), (0', \dots, 0', 3) : k = 1, 2 \} \\ &\quad \cup_{1 \leq k < l \leq n-1} \{ (i_1, \dots, i_{n-1}, 1) : i_k = i_l = 1', i_j = 0' \text{ if } j < k, i_j = 0 \text{ if } k < j < l \text{ or } l < j < n \} \\ &\quad \cup_{k=1}^{n-1} \{ (i_1, \dots, i_{n-1}, 1) : i_j = l \delta_{jk} \forall 1 \leq j \leq n-1, l = 1, 2' \} \\ &\quad \cup_{k=1}^{n-1} \{ (i_1, \dots, i_{n-1}, 2) : i_k = 1', i_j = 0' \text{ if } j < k, i_j = 0 \text{ if } k < j < n \}. \end{aligned}$$

Obviously this J_n is a subset of I_n . This proves Lemma 16. \square

Proof of (106) in the case $i_j = 3$. Let $L_0 = -\Delta + \alpha x \partial_x$. We want to show

$$\|\langle x \rangle^{-3} e^{-rL_0} P_3 |x|^3\|_{L_\infty \rightarrow L_\infty} \lesssim e^{-3\alpha r}.$$

Let $U_0(x, y)$ be the integral kernel of $U_\alpha := e^{\frac{\alpha x^2}{4}} e^{-rL_0} e^{-\frac{\alpha x^2}{4}}$. By a standard formula (see [42, 22]) we have

$$U_0(x, y) = 4\pi(1 - e^{-2\alpha r})^{-\frac{1}{2}} \sqrt{\alpha} e^{2\alpha r} e^{-\alpha \frac{(x - e^{-\alpha r} y)^2}{2(1 - e^{-2\alpha r})}}.$$

Define a new function $f := e^{-\frac{\alpha y^2}{2}} P_3 g$. The definitions above imply

$$e^{\frac{\alpha x^2}{2}} U_\alpha(\sigma + r, \sigma) e^{-\frac{\alpha x^2}{2}} P_3 g = e^{-\frac{\alpha x^2}{2}} \int U_0(x, y) f(y) dy. \quad (178)$$

Integrate by parts on the right hand side of (178) to obtain

$$e^{\frac{\alpha x^2}{2}} U_\alpha(\sigma + r, \sigma) e^{-\frac{\alpha x^2}{2}} P_3 g = e^{\frac{\alpha x^2}{2}} \int \partial_y^3 U_0(x, y) f^{(-3)}(y) dy \quad (179)$$

where $f^{(-m-1)}(x) := \int_{-\infty}^x f^{(-m)}(y) dy$ and $f^{(-0)} := f$. Because $P_3 g \perp y^m e^{-\frac{\alpha y^2}{2}}$, $m = 0, 1, 2$, we have that $f \perp 1, y, y^2$. Therefore by integration by parts we have

$$f^{(-m)}(y) = \int_{-\infty}^y f^{(-m+1)}(x) dx = - \int_y^\infty f^{(-m+1)}(x) dx, \quad m = 1, 2, 3.$$

Moreover, by the definition of $f^{(-m)}$ and the equation above we have

$$|f^{(-m)}(y)| \lesssim e^{-\frac{\alpha y^2}{2}} \langle y \rangle^{3-m} \|\langle y \rangle^{-3} P_3 g\|_\infty.$$

Using the explicit formula for $U_0(x, y)$ given above we find

$$|\partial_y^k U_0(x, y)| \lesssim \frac{e^{-\alpha k r}}{(1 - e^{-2\alpha r})^k} (|x| + |y| + 1)^k U_0(x, y).$$

We

Collecting the estimates above and using Equation (179), we have the following result

$$\begin{aligned} & \langle x \rangle^{-3} |e^{\frac{\alpha x^2}{2}} U_\alpha(\sigma + r, \sigma) e^{-\frac{\alpha x^2}{2}} P_3 g(x)| \\ & \lesssim \frac{1}{(1 - e^{-2\alpha r})^3} \langle x \rangle^{-3} e^{\frac{\alpha x^2}{2}} \int (|x| + |y| + 1)^3 e^{-3\alpha r} U_0(x, y) |f^{(-3)}(y)| dy \\ & \lesssim \frac{e^{-3\alpha r}}{(1 - e^{-2\alpha r})^3} e^{\frac{\alpha x^2}{2}} \int \langle x \rangle^{-3} U_0(x, y) e^{-\frac{\alpha}{2} y^2} \langle y \rangle^3 dy \|\langle y \rangle^{-3} P_3 g\|_\infty. \end{aligned}$$

□

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